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Optimizing nonlinear adaptive control allocation¶

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A control-Lyapunov approach is used to develop an adaptive optimizing control allocation algorithm for over-actuated mechanical systems where the actuator model is affine in the uncertain parameters. Uniform global (asymptotic) stability is guaranteed by the control allocation defined by the dynamic update laws in combination with an exponentially stable controller.

1. Introduction

Consider the system

$$\dot{x} = f(t, x, \tau) = f_1(t, x) + g(t, x)\tau \qquad (1)$$

$$\tau = h(t, x, u, \theta) = \Phi(t, x, u)\theta \qquad (2)$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $\tau \in \mathbb{R}^d$, $\theta \in \mathbb{R}^m$, and $m \leq d \leq r$. The constant parameter vector θ contains parameters of the control allocation model (actuator and effector model), that will be viewed as uncertain parameters to be adapted. Assume there exist a virtual control $\tau_c = k(t, x)$ that uniformly exponentially stabilizes the equilibrium of (1). Introducing an instantaneous cost function $J(t, x, u)$, the minimization problem

$$\min_u J(t, x, u) \quad s.t \quad \tau_c - \Phi(t, x, u)\hat{\theta} = 0 \qquad (3)$$

defines the nonlinear static control allocation problem. Since θ is an unknown parameter the idea is to use an *indirect certainty equivalence adaptive control* approach based on the estimate $\hat{\theta}$. The cost function J incorporates objectives such as minimum power consumption and input constraints (implemented as barrier functions).

Optimizing control allocation solutions have been derived for certain classes of over-actuated systems, such as for aircraft and marine vessels, (Enns, 1998), (Buffington *et al.*, 1998), (Sørdalen, 1997), (Bodson, 2002) and (Härkegård, 2002). The control allocation problem is generally viewed as a *static or quasi-dynamic* problem that is solved independently of the dynamic control problem considering non-adaptive linear models $\tau = Gu$. The main advantage of this is modularity and the ability to handle redundancy and constraints.

In the present work we consider *dynamic non-linear adaptive optimal control allocation*. Non-adaptive nonlinear control allocation has been recently studied using conventional nonlinear programming methods (Johansen *et al.*, 2004).

In (Johansen, 2004) a control Lyapunov function is used to derive an exponentially convergent dynamic update law for u (similar to a gradient/Newton-like optimization)

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such that the control allocation problem (3) is solved dynamically. It is shown that it is not necessary to solve the optimization problem (3) exactly at each time instant. It is shown that convergence and asymptotic optimality of the dynamic control allocation in combination with a uniform globally exponentially stable trajectory-tracking nonlinear controller guarantees uniform boundedness and uniform global exponential convergence of the system. One advantage of this approach is computational efficiency, since the optimizing control allocation algorithm is implemented explicitly as a dynamic nonlinear controller. Solving (3) explicitly at each sampling instant requires a computationally more expensive numerical solution of a nonlinear program to guarantee optimality. In the present work we extend the results and ideas in (Johansen, 2004) with the introduction of set-stability and adaptation in the control allocation model.

2. Parameter adaptation in the allocation equation

The first order optimality conditions for the Lagrangian

$$l(t, x, u, \lambda, \hat{\theta}) = J(t, x, u) + (\tau_c - \Phi(t, x, u)\hat{\theta})^T\lambda$$

defines local solutions to the optimizing control allocation problem (3). The design of the optimizing control allocation and adaptation laws are based on the following adaptive optimizing control Lyapunov function (aoclf)

$$V_1(t, x, \epsilon, \hat{\theta}, u, \lambda) = \sigma V_0(t, x) + \frac{1}{2}\tilde{\theta}^T Q_\theta \tilde{\theta} + \frac{1}{2}\epsilon^T Q_\epsilon \epsilon + \frac{1}{2}\left(\frac{\partial l^T}{\partial u}\frac{\partial l}{\partial u} + \frac{\partial l^T}{\partial \lambda}\frac{\partial l}{\partial \lambda}\right) \tag{4}$$

where

$$\epsilon = x - \hat{x} \tag{5}$$

$$\dot{\hat{x}} = f_1(t, x) + A_\epsilon(x - \hat{x}) + g(t, x)\Phi(t, x, u)\hat{\theta} \tag{6}$$

$\sigma > 0$ is an arbitrary constant, $\tilde{\theta} = \theta - \hat{\theta}$ and the design matrices satisfies $A_\epsilon = A_\epsilon^T > 0$, $Q_\theta = Q_\theta^T > 0$ and $Q_\epsilon = Q_\epsilon^T > 0$. The first term in (4) contains the Lyapunov function inherited from the exponential stable virtual controller:

Assumption 1. There exists a differentiable function $V_0 : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and positive constants c_1, c_2, c_3, c_4 such that $\forall t \geq 0$

$$c_1 \|x\|^2 \leq V_0(t, x) \leq c_2 \|x\|^2$$

$$\frac{\partial V_0}{\partial t}(t, x) + \frac{\partial V_0^T}{\partial x}(t, x) f(t, x, k(t, x)) \leq -c_3 \|x\|^2$$

$$\left\|\frac{\partial V_0}{\partial x}(t, x)\right\| \leq c_4 \|x\|$$

The last term in (4) incorporates the first order optimality condition for the Lagrangian as in (Johansen, 2004). The second term is standard extension of the Lyapunov function for the certainty equivalence approach (Krstic *et al.*, 1995), while the third term is introduced to make $\hat{\tau} = \Phi\hat{\theta} \to \tau$, such that $\tau \to \tau_c$ which will be shown to support the convergence of $\tilde{\theta} \to 0$ as $t \to \infty$. The time derivative of V_1 along the trajectories of the system is given by

$$\dot{V}_1 = \sigma\left(\frac{\partial V_0}{\partial t}(t, x) + \frac{\partial V_0^T}{\partial x}(t, x) f(t, x, k(t, x))\right)$$

$$
\begin{aligned}
&+\alpha^T\dot{u}+\beta^T\dot{\lambda}+\delta-\epsilon^T Q_\epsilon A_\epsilon \epsilon \\
&+\left(\frac{\partial l^T}{\partial u}\frac{\partial^2 l}{\partial x \partial u}+\frac{\partial l^T}{\partial \lambda}\frac{\partial^2 l}{\partial x \partial \lambda}\right) g(t,x)\Phi(t,x,u)\tilde{\theta} \\
&+\epsilon^T Q_\epsilon g(t,x)\Phi(t,x,u)\tilde{\theta}+\tilde{\theta}^T Q_\theta \dot{\tilde{\theta}} \\
&+\sigma\frac{\partial V_0^T}{\partial x}(t,x)g(t,x)(\Phi(t,x,u)\theta-\tau_c)
\end{aligned} \tag{7}
$$

where

$$
\beta=-\frac{\partial(\Phi\hat{\theta})}{\partial u}\frac{\partial l}{\partial u} \tag{8}
$$

$$
\begin{aligned}
\delta=&\frac{\partial l^T}{\partial u}\frac{\partial^2 l}{\partial t \partial u}+\left(\frac{\partial l^T}{\partial u}\frac{\partial^2 l}{\partial \hat{\theta} \partial u}+\frac{\partial l^T}{\partial \lambda}\frac{\partial^2 l}{\partial \hat{\theta} \partial \lambda}\right)\dot{\hat{\theta}} \\
&+\left(\frac{\partial l^T}{\partial u}\frac{\partial^2 l}{\partial x \partial u}+\frac{\partial l^T}{\partial \lambda}\frac{\partial^2 l}{\partial x \partial \lambda}\right)f(t,x,\tau)+\frac{\partial l^T}{\partial \lambda}\frac{\partial^2 l}{\partial t \partial \lambda}
\end{aligned} \tag{9}
$$

$$
\alpha=\frac{\partial^2 l}{\partial u^2}\frac{\partial l}{\partial u}-\frac{\partial(\Phi\hat{\theta})^T}{\partial u}\frac{\partial l}{\partial \lambda} \tag{10}
$$

We will show that optimization and adaptation laws can be designed by the aoclf. For system (1–2) we propose the following parameter update law

$$
\dot{\hat{\theta}}=Q_\theta^{-1}\Phi_\theta^T\Gamma_\theta^T \tag{11}
$$

$$
\Gamma_\theta=\left(\frac{\partial l^T}{\partial u}\frac{\partial^2 l}{\partial x \partial u}+\frac{\partial l^T}{\partial \lambda}\frac{\partial^2 l}{\partial x \partial \lambda}+\epsilon^T Q_\epsilon+\sigma\frac{\partial V_0^T}{\partial x}\right)
$$

$$
\Phi_\theta=g(t,x)\Phi(t,x,u)
$$

the certainty equivalent control allocation update laws

$$
\dot{u}=-\Gamma\alpha+\zeta+\zeta_0 \tag{12}
$$

$$
\dot{\lambda}=-W\beta+\phi+\phi_0 \tag{13}
$$

with $\Gamma=\Gamma^T>0$, $W=W^T>0$. ζ and ϕ satisfy the algebraic equation

$$
\alpha^T\zeta+\beta^T\phi+\delta_0=0 \tag{14}
$$

where $\delta_0=\delta+\alpha^T\zeta_0+\beta^T\phi_0$, and ζ_0 and ϕ_0 are defined by

$$
(\zeta_0;\phi_0)=-\mathbb{H}^{-1}\left(\frac{\partial^2 l}{\partial t \partial u};\frac{\partial^2 l}{\partial t \partial \lambda}\right) \tag{15}
$$

where

$$
\mathbb{H}=\begin{pmatrix} \dfrac{\partial^2 \ell}{\partial u^2} & -\left(\dfrac{\partial \Phi}{\partial u}\hat{\theta}\right)^T \\ -\left(\dfrac{\partial \Phi}{\partial u}\hat{\theta}\right) & 0 \end{pmatrix}.
$$

For the purpose of analyzing the prospects of the above control allocation, we write the closed loop dynamics in the compact form

$$\dot{z} = F(z) \tag{16}$$

where $F(z)$ is given by (1–2), (5), (11–13), $\dot{p} = 1$, $p_0 = t_0$, $z = (p; x; \epsilon; \hat{\theta}; u; \lambda)$, (;) is a column-stacking operator and p is the time-state. By introducing the optimal set

$$\mathcal{A} = \{z \in \mathbb{R}^q \mid G(z) = 0\} \tag{17}$$

where,

$$G(z) = \left(x; \epsilon; \tilde{\theta}; \frac{\partial l}{\partial u}; \frac{\partial l}{\partial \lambda}\right),$$

the set-stability analysis can be done in the same way as for a time-invariant model.

We present the concept of set-stability through definitions 1–6 from (Teel *et al.*, 2002).

Definition 1. The distance from a point $z \in \mathbb{R}^q$ to the set $\mathcal{A} \subset \mathbb{R}^q$ is defined by

$$|z|_{\mathcal{A}} = \inf\{d(z, y) \mid y \in \mathcal{A}\} \tag{18}$$

where $d(z, y)$ can be any metric.

Definition 2. The system (16) is said to be *forward complete* if, for each $z_0 \in \mathbb{R}^q$ the solution $z(t, z_0)$ is defined on $[0, \infty)$.

Definition 3. A nonempty closed set $\mathcal{A} \subset \mathbb{R}^q$ is a *forward invariant* set for (16) if the system is forward complete and $\forall z_0 \in \mathcal{A}$ the solution $z(t, z_0) \in \mathcal{A}$, $\forall t \geq 0$.

Definition 4. The system (16) is said to be *finite escape time detectable* through $|\cdot|_{\mathcal{A}}$, if a solution $z(t, z_0)$ is right maximally defined on a bounded interval $[0, T)$, then $\lim_{t \nearrow T} |z(t, z_0)|_{\mathcal{A}} = \infty$.

Definition 5. For the system (16), the closed set $\mathcal{A}$ is Uniformly Globally Stable (UGS) if the system (16) is forward complete and there exists $\rho \in \mathcal{K}_\infty$ such that, $\forall\, z_0 \in \mathbb{R}^q$

$$|z(t, z_0)|_{\mathcal{A}} \leq \rho(|z_0|_{\mathcal{A}}), \ \forall t \geq 0$$

Definition 6. For the system (16), the closed set $\mathcal{A}$ is Uniformly Globally Asymptotically Stable (UGAS) if it is UGS and for each R, $\varepsilon > 0$ there exists a $T(R, \varepsilon) > 0$ such that, $\forall z_0 \in \mathbb{R}^q$

$$|z_0|_{\mathcal{A}} \leq R, \quad t \geq T \Rightarrow |z(t, z_0)|_{\mathcal{A}} \leq \varepsilon$$

Definition 7. A smooth Lyapunov function for (16) with respect to a non-empty, closed forward invariant set $\mathcal{A} \subset \mathbb{R}^q$ is a function $V: \mathbb{R}^q \to \mathbb{R}$ that satisfies: (i) there exists two $\mathcal{K}_\infty$ functions α_1 and α_2 such that for any $z \in \mathbb{R}^q$, $\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}})$. (ii) There exists a continuous and positive semidefinite function α_3 such that for any

$$z \in \mathbb{R}^q \setminus \mathcal{A}: \frac{dV}{dz} F(z) \leq -\alpha_3(|z|_{\mathcal{A}}).$$

Theorem 1. Assume system (16) is finite escape-time detectable through $|z|_{\mathcal{A}}$. If there exists a smooth Lyapunov function for the system (16) with respect to a nonempty, closed, forward invariant set $\mathcal{A}$, then $\mathcal{A}$ is UGS with respect to (16).

Definition 7 and Theorem 1 are found in (Skjetne, 2005).

Assumption 2. There exists constants ϱ_2, $\varrho_1 > 0$ such that $\forall t$, x and u.

$$\varrho_1 I_d \leq \frac{\partial \Phi}{\partial u}(t, x, u)\frac{\partial \Phi^T}{\partial u}(t, x, u) \leq \varrho_2 I_d \tag{19}$$

Assumption 3. The function f is differentiable and satisfies $f(t, 0, 0) = 0$. Moreover, it is globally Lipschitz, uniformly in t with Lipschitz constants L_x and L_τ in x and τ. The function Φ is twice differentiable and globally Lipschitz, uniformly in t, with $\Phi(t, 0, 0) = 0$ and Lipschitz constant L_Φ in x and u. The function k is differentiable and Lipschitz in x, uniformly in t, with $k(t, 0) = 0$.

Assumption 4. The cost function J is twice differentiable.

Assumption 5. There exists constants $k_2 > k_1 > 0$ such that $\forall t$, x, u, λ and $\hat{\theta}$

$$k_1 I_r < \frac{\partial^2 l}{\partial u^2}(t, x, u, \lambda, \hat{\theta}) < k_2 I_r \tag{20}$$

Assumption 6. For all t, x, and $\hat{\theta}$, the set

$$\left\{u, \lambda \in \mathbb{R}^{r+d} \middle| \left(\frac{\partial l}{\partial u}; \frac{\partial l}{\partial \lambda}\right)(z) = 0\right\}$$

is bounded.

Claim 1. The set $\mathcal{A}$ is a closed and forward invariant set for the system (16).

Proof. From Proposition 1.1.9 (b) in (Bertsekas *et al.*, 2003) we have that $G : \mathbb{R}^q \to \mathbb{R}^{q_G}$ is continuous iff $G^{-1}(U)$ is closed in $\mathbb{R}^q$ for every closed U in $\mathbb{R}^{q_G}$. From the definition of $\mathcal{A}$, $U = \{0\}$, and since G is continuous (by assumption 2–5), $\mathcal{A}$ is a closed set. The set is forward invariant if at t_1, $G(z(t_1)) = 0$ and

$$\frac{d(G(z(t)))}{dt} = 0 \ \forall t > t_1$$

with respect to (16). We have $G(z(t_1)) = 0 \Rightarrow (\dot{x}, \dot{\epsilon}, \dot{\hat{\theta}}) = 0$, by assumption 3 and equations (1–2, 5 and 11), and $(\alpha, \beta, \zeta, \phi) = 0$ from (10–9). It remains to prove

$$\left(\frac{d\partial l}{dt\partial u}; \frac{d\partial l}{dt\partial \lambda}\right) = 0.$$

We get

$$\frac{d\partial l}{dt\partial \lambda} = \frac{\partial^2 l}{\partial t\partial \lambda} + \frac{\partial^2 l}{\partial x\partial \lambda}\dot{x} + \frac{\partial^2 l}{\partial u\partial \lambda}\dot{u} + \frac{\partial^2 l}{\partial \hat{\theta}\partial \lambda}\dot{\hat{\theta}}$$

and

$$\frac{d\partial l}{dt\partial u} = \frac{\partial^2 l}{\partial t\partial u} + \frac{\partial^2 l}{\partial x\partial \lambda}\dot{x} + \frac{\partial^2 l}{\partial u\partial u}\dot{u} + \frac{\partial^2 l}{\partial \lambda\partial u}\dot{\lambda} + \frac{\partial^2 l}{\partial \hat{\theta}\partial u}\dot{\hat{\theta}},$$

thus from the arguments above

$$\left(\frac{d\partial l}{dt\partial u}; \frac{d\partial l}{dt\partial \lambda}\right) = \mathbb{H}\left(\dot{u}; \dot{\lambda}\right) + \left(\frac{\partial^2 l}{\partial t\partial u}; \frac{\partial^2 l}{\partial t\partial \lambda}\right)$$

then by inserting control law (12–13) we get

$$\left(\frac{d\partial l}{dt\partial u}; \frac{d\partial l}{dt\partial \lambda}\right) = 0$$

and $G(z(t, z(t_1))) = 0$ for all $t > t_1$, which proves the claim. ■

Claim 2. The system (16) is finite escape-time detectable through $|z|_{\mathcal{A}}$.

Proof. $G^{-1}(0)$ is bounded by assumption 6 and $\|\theta\|_\infty < \infty$, and since all states except the time-state p are represented linearly in (17), the system (16) is finite escape time detectable through $|z|_{\mathcal{A}}$. ■

Claim 3. There exist positive constants κ_1, κ_2 such that

$$\begin{aligned} &\kappa_1(\|\tilde{u}\|^2 + \|\tilde{\lambda}\|^2 + \|\tilde{\theta}\|^2) \\ &\leq \left(\frac{\partial l^T}{\partial u}\frac{\partial l}{\partial u} + \frac{\partial l^T}{\partial \lambda}\frac{\partial l}{\partial \lambda} + \tilde{\theta}^T Q_0 \tilde{\theta}\right) \\ &\leq \kappa_2(\|\tilde{u}\|^2 + \|\tilde{\lambda}\|^2 + \|\tilde{\theta}\|^2) \end{aligned} \tag{21}$$

Proof. The result follows by applying the same procedure as in proof of Proposition 1 in (Johansen, 2004). ■

The main results are summarized in the following propositions.

Proposition 1. Consider the system (1–2), with the update-laws (11–13), then

(i) The algebraic equation (14) is always solvable, and there exists a unique solution for ζ and ϕ.
(ii) There exists class $\mathcal{K}_\infty$ functions α_1 and α_2 s.t $\alpha_1(|z|_{\mathcal{A}}) \leq V_1(z) \leq \alpha_2(|z|_{\mathcal{A}})$
(iii) The set $\mathcal{A}$ is rendered UGS and

$$\left(x; \epsilon; \frac{\partial l}{\partial \lambda}; \frac{\partial l}{\partial u}\right) \to 0 \text{ as } t \to \infty$$

Proof.

(i) This follows from ($\alpha = 0$ and $\beta = 0$) $\Leftrightarrow$ ($\delta_0 = 0$) by lemma 1 and 2 in (Johansen, 2004).
(ii) This follows from claim 3 and assumption 1.
(iii) By inserting the update laws (11–13) and the algebraic equation (14) into (7), we get

$$\dot{V}_1 = \sigma\left(\frac{\partial V_0}{\partial t}(t, x) + \frac{\partial V_0^T}{\partial x}(t, x) f(t, x, \tau_c)\right)$$

$$
- \alpha^T \Gamma \alpha - \beta^T W \beta - \epsilon^T Q_\epsilon A_\epsilon \epsilon
$$
$$
+ \sigma \frac{\partial V_0^T}{\partial x}(t, x) g(t, x)(\Phi(t, x, u)\hat{\theta} - \tau_c) \tag{22}
$$

then by following the same procedure as in (Johansen, 2004)

$$
\dot{V}_1 \leq -\sigma(c_3 - M\mu)\|x\|^2 - \left(\lambda_{\min}(\Gamma) - \frac{\sigma M}{\mu}\right)\|\alpha\|^2
$$
$$
- \left(\lambda_{\min}(W) - \frac{\sigma M}{\mu}\right)\|\beta\|^2 - \lambda_{\min}(Q_\epsilon A_\epsilon)\|\epsilon\|^2
$$

where

$$
M = \max\left(\frac{L_\tau L_{\Phi c_4}}{\varrho_1}, \frac{L_\tau L^2_{\Phi c_4 \kappa_2}}{\varrho_1^2}\right),
$$

thus $\mu > 0$ and $\sigma > 0$ can be chosen such that there exist positive constants k_3, k_4, k_5 and k_6 satisfying

$$
\dot{V}_1 \leq -k_3\|x\|^2 - k_4\left\|\frac{\partial l}{\partial \lambda}\right\|^2 - k_5\left\|\frac{\partial l}{\partial u}\right\|^2 - k_6\|\epsilon\|^2 = -\alpha_3(|z|_{\mathcal{A}}) \leq 0 \tag{23}
$$

With V_1 as the Lyapunov function candidate Theorem 1 is satisfied and the UGS property is established. Thus $G(z(t)) \in \mathcal{L}_\infty$. The convergence result follows from

$$
\lim_{t\to\infty} \int_{t_0}^{t} \alpha_3(|z(s)|_{\mathcal{A}})ds \leq \lim_{t\to\infty} \int_{t_0}^{t} -\dot{V}_1(z(s))ds \leq V_1(z(t_0)) < \infty \tag{24}
$$

and

$$
\left(x;\ \epsilon;\frac{\partial l}{\partial \lambda};\frac{\partial l}{\partial u}\right)^T\left(x;\ \epsilon;\frac{\partial l}{\partial \lambda};\frac{\partial l}{\partial u}\right) \leq K\alpha_3(|z|_{\mathcal{A}}) \Rightarrow \left(x;\ \epsilon;\frac{\partial l}{\partial \lambda};\frac{\partial l}{\partial u}\right) \in \mathcal{L}_2
$$

where

$$
K = \frac{1}{\min(k_3, k_4, k_5, k_6)}.
$$

By the assumptions we also have $\dot{G} \in \mathcal{L}_\infty$ since $|z|_{\mathcal{A}} \in \mathcal{L}_\infty$. Thus according to Barbalat's lemma,

$$
\left(x;\ \epsilon;\frac{\partial l}{\partial \lambda};\frac{\partial l}{\partial u}\right) \to 0 \text{ as } t \to \infty. \qquad \blacksquare
$$

Definition 8. A piecewise continuous signal matrix $\Phi : \mathbb{R} \to \mathbb{R}^{n\times m}$ is Persistently Excited (PE) if there exists constants γ, $\Delta > 0$, such that

$$
\frac{1}{\Delta}\int_t^{t+\Delta} \Phi^T(\tau)\Phi(\tau)dr \geq \gamma I_{m\times m},\ \forall t > t_0 \tag{25}
$$

Claim 4.

$$
\lim_{t\to\infty} \int_{t_0}^{t} \tilde{\theta}^T\left(\Phi_0^T\Phi_0 - \frac{\mu_1}{2}I_{m\times m}\right)\tilde{\theta}dt < \infty \forall \mu_1 > 0.
$$

Proof. From Proposition 1 we have $\epsilon \in \mathcal{L}_2$ and

$$\int \dot{\epsilon}^T \dot{\epsilon} dt = \epsilon^T \dot{\epsilon} + \int \epsilon^T A_\epsilon \epsilon dt - \int \epsilon^T \dot{\Phi}_\theta \tilde{\theta} dt + \int \epsilon^T \Phi_\theta \Phi_\theta^T Q_\theta^{-T} \Gamma_\theta^T dt \tag{26}$$

$$\lim_{t\to\infty} \int_{t_0}^{t} 2\epsilon^T A_\epsilon \dot{\epsilon} dt < \infty \tag{27}$$

$$\lim_{t\to\infty} \int_{t_0}^{t} \| \epsilon^T \Phi_\theta \Phi_\theta^T Q_\theta^{-1} \Gamma_\theta^T \| dt < \infty \tag{28}$$

Integral (28) follows by inequality

$$2\| \epsilon^T \Phi_\theta \Phi_\theta^T Q_\theta^{-1} \Gamma_\theta^T \| \leq \| \epsilon^T \Phi_\theta \Phi_\theta^T Q_\theta^{-1} Q_\epsilon^T \epsilon \| + \left\| \sigma \frac{\partial V_0^T}{\partial x} \right\|^2 + \left\| \frac{\partial l^T}{\partial u} \frac{\partial^2 l}{\partial x \partial u} \right\|^2 + \left\| \frac{\partial l^T}{\partial \lambda} \frac{\partial^2 l}{\partial x \partial \lambda} \right\|^2$$

In addition to the integrals from above we will use

$$-\int \epsilon^T \dot{\Phi}_\theta \tilde{\theta} dt \leq \frac{1}{2\mu_1} \int (\epsilon^T \dot{\Phi}_\theta \dot{\Phi}_\theta^T \epsilon)\, dt + \frac{\mu_1}{2} \int \tilde{\theta}^T \tilde{\theta} dt$$

which follows from *Young's inequality*, $\forall \mu_1 > 0$.

$$\lim_{t\to\infty} \int_{t_0}^{t} \tilde{\theta}^T \Phi_\theta^T \Phi_\theta \tilde{\theta} dt = \lim_{t\to\infty} \int_{t_0}^{t} (\dot{\epsilon}^T_\epsilon + 2\epsilon^T A_\epsilon \dot{\epsilon} + \epsilon^T \epsilon)\, d\tau$$

$$\leq \lim_{t\to\infty} \int_{t_0}^{t} \left(\frac{1}{2\mu_1} \epsilon^T \dot{\Phi}_\theta \dot{\Phi}_\theta^T \epsilon + \frac{\mu_1}{2} \tilde{\theta}^T \tilde{\theta} \right) d\tau + K_\epsilon$$

where $K_\epsilon > 0$ defined by (27–28), thus

$$\lim_{t\to\infty} \int_{t_0}^{t} \tilde{\theta}^T \left(\Phi_\theta^T \Phi_\theta - \frac{\mu_1}{2} I_{m\times m} \right) \tilde{\theta} dt \leq \frac{1}{2\mu_1} \lim_{t\to\infty} \int_{t_0}^{t} (\epsilon^T \dot{\Phi}_\theta \dot{\Phi}_\theta^T \epsilon)\, d\tau + K_\epsilon < \infty$$

since $\dot{\Phi}_\theta \in \mathcal{L}_\infty$, the claim is proved. ■

Proposition 2. If $\Phi_\theta(t)$ is PE and the results from Proposition 1 holds, then $\mathcal{A}$ is UGAS.

Proof. From Proposition 1 and definition 5, we have

$$|z(t, z_0)|_{\mathcal{A}} \leq \rho(|z_0|_{\mathcal{A}}),\ \forall t \geq t_0 \tag{29}$$

where $\rho \in \mathcal{K}_\infty$. Fix $R > 0$ and $\varepsilon > 0$. Define $\Omega = \rho(R)$, $\omega = \min\{\Omega, \rho^{-1}(\varepsilon)\}$ and

$$\xi(z) = G(z)^T G(z) - \tilde{\theta}^T \tilde{\theta} + \tilde{\theta}^T \left(\Phi_\theta^T \Phi_\theta - \frac{\mu_1}{2} I_{m\times m} \right) \tilde{\theta}$$

Note from claim 2 and Proposition 1, that

$$\int_{t_0}^{t_0+T} \xi(z(\tau, z_0))\, d\tau \leq \lim_{t\to\infty} \int_{t_0}^{t} \xi(z(\tau, z_0))\, d\tau \leq B \tag{30}$$

where B is given by ω and Ω. Define

$$T = \frac{2B}{\gamma_1 \omega_1},$$

where ω_1 and γ_1 are specified later, and assume that $\forall |z_0|_{\mathcal{A}} \leq R$ there exists $t' \in [t_0, T]$ such that $|z(t', z_0)|_{\mathcal{A}} \leq \rho^{-1}(\varepsilon)$. Thus $|z(t, z_0)|_{\mathcal{A}} \leq \rho(|z(t', z_0)|_{\mathcal{A}}) \leq \rho(\rho^{-1}(\varepsilon)) = \varepsilon$ for $|z_0|_{\mathcal{A}} \leq R$ and $t \geq T + t_0$, which satisfies definition 6. Suppose this assumption is not true. i.e., there exists $|z_0|_{\mathcal{A}} \leq R$ such that $|z(t', z_0)|_{\mathcal{A}} > \rho^{-1}(\varepsilon) \forall t' \in [t_0, T]$. Thus

$$\omega \leq |z(t', z_0)|_{\mathcal{A}} \leq \Omega \forall t' \in [t_0, T]$$

which from (21) imply that there exist positive constants ω_1, and Ω_1 such that

$$\omega_1 \leq G(z(t', z_0))^T G(z(t', z_0)) \leq \Omega_1 \ \forall t' \in [t_0, T]$$

By introducing $\mathcal{M} = \{1, 2, \ldots, q\}$, and $\mathcal{I}(t_s) = \arg\max_i |G^i(z(t_s, z_0))|$, we can construct a new vector G_ε, by $G_\varepsilon^{\mathcal{I}(t_s)}(z(t_s, z_0)) = G^{\mathcal{I}(t_s)}(z(t_s, z_0))$ and $G_\varepsilon^{\mathcal{M}\setminus\mathcal{I}(t_s)}(z(t_s, z_0)) = 0$. We use $i = \mathcal{I}(t_s)$ and since

$$\|G(z(t', z_0))\|_\infty \geq \sqrt{\frac{\omega_1}{q}}, \forall t' \in [t_0, T]$$

we have

$$|G_\varepsilon^i(z(t_s, z_0))| > \sqrt{\frac{\omega_1}{q}}.$$

Since G is uniformly continuous, there is a positive constant

$$t_{s+1}\left(\sqrt{\frac{\omega_1}{q}}\right)$$

such that

$$|G_\varepsilon^i(z(t_s, z_0)) - G_\varepsilon^i(z(t, z_0))| < \sqrt{\frac{\omega_1}{4q}} \forall t_s > t_0$$

and $\forall t > t_0$ with

$$|t_s - t| \leq t_{s+1}\left(\sqrt{\frac{\omega_1}{q}}\right).$$

Hence, $\forall t \in [t_s, t_s + t_{s+1}]$ we have

$$\begin{aligned} |G_\varepsilon^i(z(t, z_0))| &= |G_\varepsilon^i(z(t, z_0)) - G_\varepsilon^i(z(t_s, z_0)) + G_\varepsilon^i(z(t_s, z_0))| \\ &\geq |G_\varepsilon^i(z(t_s, z_0))| - |G_\varepsilon^i(z(t_s, z_0)) - G_\varepsilon^i(z(t_s, z_0))| \\ &> \sqrt{\frac{\omega_1}{q}} - \sqrt{\frac{\omega_1}{4q}} = \sqrt{\frac{\omega_1}{4q}} \end{aligned} \tag{31}$$

which implies that

$$\begin{aligned} \int_{t_s}^{t_{s+1}} \xi(z(t, z_0))\,dt &= \int_{t_s}^{t_{s+1}} \left(G(z(t, z_0))^T A_G G(z(t, z_0)) - \frac{\mu_1 \tilde{\theta}^T \tilde{\theta}}{2} \right) dt \\ &\geq \int_{t_s}^{t_{s+1}} \left(G_\varepsilon(z(t, z_0))^T A_G G_\varepsilon(z(t, z_0))\,dt - \frac{\mu_1 \Omega_1}{2} \right) dt \end{aligned}$$

$$\geq \left(\frac{1}{4q}\omega_1 \min(\gamma, 1) - \frac{\mu_1\Omega_1}{2}\right)(t_{s+1} - t_s) \tag{32}$$

where

$$A_G = \begin{pmatrix} I_{2n\times 2n} & 0 & 0 \\ 0 & \Phi_0^T\Phi_0 & 0 \\ 0 & 0 & I_{(r+d)\times(r+d)} \end{pmatrix},$$

$$\gamma_1 = \left(\frac{\omega_1 \min(\gamma, 1)}{4q} - \frac{\mu_1\Omega_1}{2}\right)$$

and μ_1 is chosen to keep γ_1 strictly positive. Thus

$$\int_{t_0}^{t_0+T} \xi(z(t, z_0))\,dt > \sum_{s=t_0}^{t_0+T} \gamma_1\omega_1 (t_{s+1} - t_s) = 2B$$

which is a contradiction to (30). Since $\xi(z(t, z_0)) \to 0$ uniformly then from (21) we have that $\mathcal{A}$ is uniformly globally attractive, and consequently by applying the UGS property from Proposition 1, $\mathcal{A}$ is UGAS. ■

Remark 1. If τ is known (e.g. measured using accelerometers), the θ can be estimated directly from the allocation model $\tau = \Phi(t, x, u)\theta$.

Remark 2. If

$$\Gamma_0 = \left(\frac{\partial l^T}{\partial u}\frac{\partial^2 l}{\partial x \partial u} + \frac{\partial l^T}{\partial \lambda}\frac{\partial^2 l}{\partial x \partial \lambda} + \epsilon^T Q_\epsilon\right)$$

is used in the update law, then

$$\dot{V}_1 \leq -c_3\sigma \|x\|^2 - \alpha^T\Gamma\alpha - \beta^T W\beta - \epsilon^T Q_\epsilon A_\epsilon \epsilon + 2\sigma L_\tau c_4 \|x\| \left\|\frac{\partial l}{\partial \lambda}\right\|$$

$$+ 2\sigma L_\tau c_4 \|x\| (L_{\Phi u}\|u\| + L_{\Phi x}\|x\|)\|\tilde{\theta}\| \tag{33}$$

and some local stability properties, dependent on the system and virtual controller, may be concluded (the proof is not included in this paper, but the result is shown in the simulation example).

Remark 3. The matrices $\Gamma > 0$ and $W > 0$ may be time-varying, without changing any theoretical properties, provided they are bounded away from zero. Newton-like methods can therefore be implemented by taking

$$(\dot{u}; \dot{\lambda}) = -\gamma(\mathbb{H}^T\mathbb{H} + \varepsilon I_{r+p})^{-1}(\alpha; \beta) + (\zeta; \phi) + (\zeta_0; \phi_0) \tag{34}$$

where $\gamma > 0$, $\varepsilon \geq 0$ and

$$(\alpha; \beta) = \mathbb{H}\left(\frac{\partial \ell}{\partial u}; \frac{\partial \ell}{\partial \lambda}\right)$$

are time-varying.

3. Simulation example

We consider the case study of manœuvering a low-speed and overactuated ship where the unknown parameters in the allocation model represents thrust loss. The non-adaptive version of this example was found in (Johansen, 2004). The example is based on (Lindegaard and Fossen, 2003) and the scaled-model ship dynamics are given by

$$\begin{aligned} \dot{\eta}_e &= R(\psi)\nu \\ \dot{\nu} &= -M^{-1}D\nu + M^{-1}(\tau + b) \\ \tau &= \Phi(u)\theta \end{aligned} \tag{35}$$

and the augmented integral action $\dot{\xi}_i = \eta_e$. The position $\eta_e = (x_e;\ y_e;\ \psi_e) = (x - x_d;\ y - y_d;\ \psi - \psi_d)$ is the north, east positions and compass heading deviations. Subscript d denotes the desired state. $\nu = (u;\ v;\ r)$ is the body-fixed velocities, surge, sway and yaw, τ is the generalized force vector, $b = (b_1;\ b_2;\ b_3)$ is a bias due to wind and current and $R(\psi)$ the rotation matrix function between body fixed and earth fixed coordinate frame. In the considered model there are five actuators; the two main propellers aft of the hull, in conjunction with two rudders, and one tunnel thruster going through the hull of the vessel. ω_i denotes the propeller velocity and δ_i denotes the rudder deflection. $i = 1, 2$ denotes the aft actuators, while $i = 3$ denotes the tunnel thruster. This model can be rewritten in the form (1) and (2) by:

$$x = (\xi;\ \eta_e;\ \nu),\ \theta = (\theta_1;\ \theta_2)$$

$$\tau = (\tau_1;\ \tau_2;\ \tau_3),\ u = (\omega_1;\ \omega_2;\ \omega_3;\ \delta_1;\ \delta_2)$$

$$\Phi(u) = \begin{pmatrix} X_1 + X_2 & 0 \\ Y_1 + Y_2 & Y_3 \\ \Phi_{13} & l_{3,x}Y_3 \end{pmatrix}$$

$$\Phi_{13} = -l_{1,y}X_1 + l_{1,x}Y_1 - l_{2,y}X_2 + l_{2,x}Y_2$$

$$X_i = T_i - D_i, \quad Y_i = L_i$$

where the propulsion forces are defined by

$$T_i = \begin{cases} k_{Tp_i}\omega_i^2 & \omega_i \geq 0 \\ k_{Tn_i}\,|\,\omega_i\,|\,\omega_i & \omega_i < 0 \end{cases}$$

$$L_i = \begin{cases} T_i(1 + k_{Ln_i}\omega_i)(k_{L\delta 1_i} + k_{L\delta 2_i}\,|\,\delta_i\,|\,)\delta_i, & \omega_i \geq 0 \\ 0, & \omega_i < 0 \end{cases}$$

$$D_i = \begin{cases} T_i(1 + k_{Dn_i}\omega_i)(k_{D\delta 1_i}\,|\,\delta_i\,| + k_{D\delta 2_i}\delta_i^2), & \omega \geq 0 \\ 0, & \omega < 0 \end{cases}$$

The unknown parameter vector θ represents thrust loss. θ_2 is also related to the parameters k_{Tp_3} and k_{Tn_3} in a multiplicative way. This suggest that the estimate of θ_2 gives a direct estimate of the tunnel thruster parameter. A virtual controller τ_c that stabilizes the system (35) uniformly, globally and exponentially for some physically limited yaw rate is proposed in (Lindegaard and Fossen, 2003) and given by

$$\tau_c = -K_iR^T(\psi)\xi - K_pR^T(\psi)\eta_e - K_d\nu \tag{36}$$

The cost function used in this simulation is the same as was used in (Johansen, 2004)

$$J(u) = \sum_{i=1}^{3} k_i \mid \omega_i \mid \omega_i^2 + \sum_{i=1}^{2} q_i \delta_i^2$$

$$\mid \omega_i \mid \leq 18Hz, \quad \mid \delta_i \mid \leq 35 \text{ deg}$$

$$k_1 = k_2 = 0.001, \quad k_3 = 0.02, \quad q_1 = q_2 = 500$$

Consider a wind disturbance vector $b = 0.05(1;\ 1;\ 1)$, the design matrix $A_\epsilon = I_{9\times 9}$, the true parameter vector $\theta = (1;\ 1)^T$, its update gain matrix $Q_\theta = diag(1;\ 1)$ and the ϵ error weight $Q_\epsilon = diag(a,\ 10^5;\ 2\cdot 10^5;\ 2\cdot 10^4)$, $a = 10^3\ (1;\ 1;\ 1;\ 1;\ 1;\ 1)$, and the parameter update law from remark 2. The implementation of Γ and W was done according to remark 3 where $\gamma = 1$ and $\varepsilon = 10^{-9}$. The simulation results are presented in the Figures 1–4. At $t_1 \approx 200$ and $t_2 \approx 400$ the parameter update law are excited and the estimated parameters converges to the true values. For different initial conditions, it can be shown that $\hat{\theta} \not\to \theta$, since Φ may not be *PE* over a sufficient timespan for the reference signals. With white noise perturbations or harmonic references, Φ can be shown to be *PE* for all t such that $\hat{\theta} \to \theta$. This is verified by simulations.

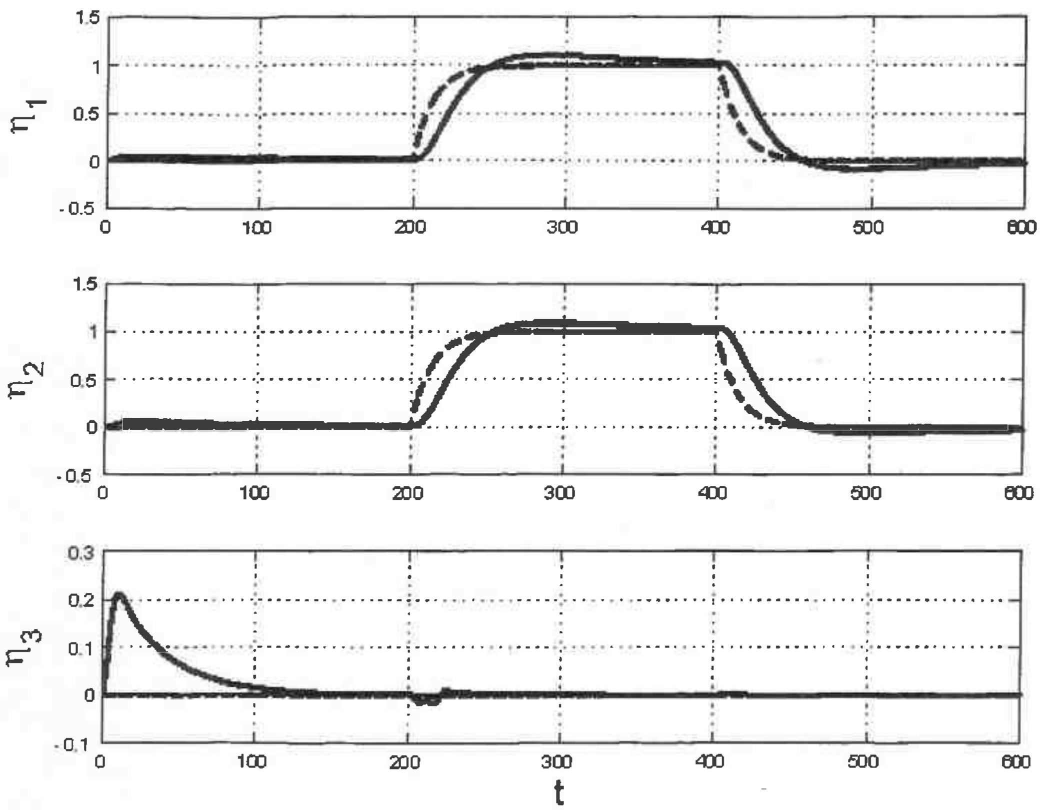

Figure 1. Simulation results—the solid lines represents positions while the dashed lines represents references

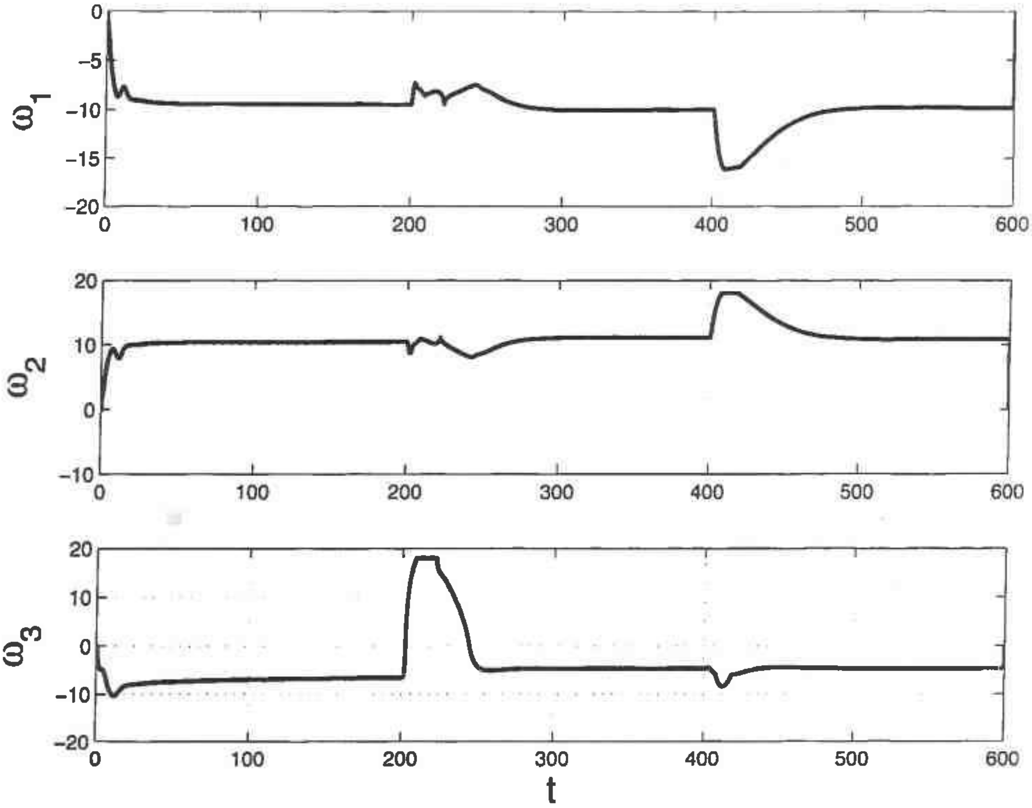

Figure 2. Simulation results—computed propeller velocities by the allocation algorithm

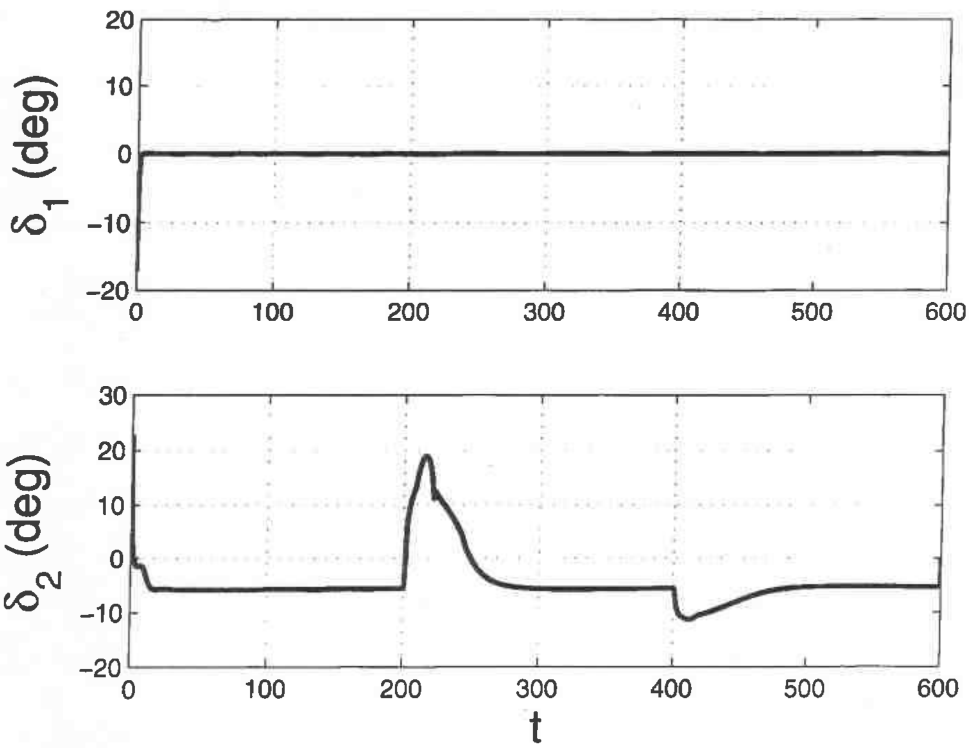

Figure 3. Simulation results—computed rudder deflection by the allocation algorithm

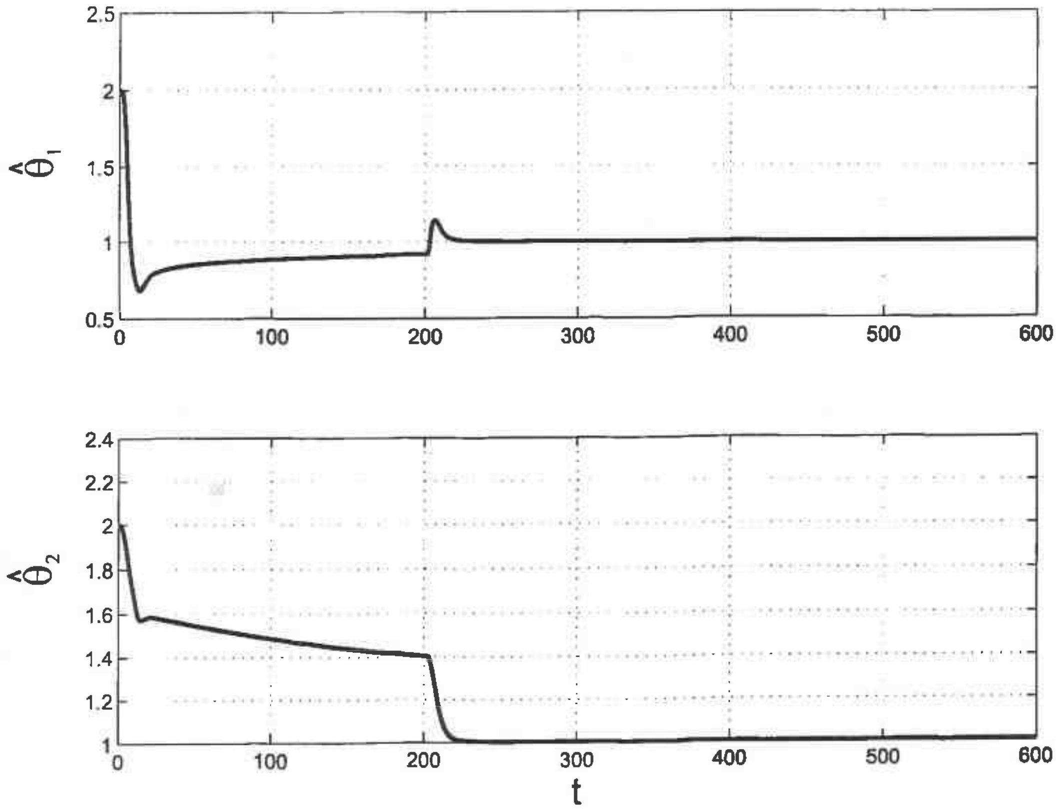

Figure 4. Simulation results—the parameter adaption

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