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The Riccati Equation — An Economic Fundamental Equation which Describes Marginal Movement in Time

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The objective of this article is to demonstrate that the Riccati equation is an economic fundamental equation, which is marginally descriptive in time for the large majority of economic systems.

It is also an objective to interpret the Riccati equation and the corresponding relations in terms of economics.

The article shows that there is a close relationship between the marginal Hamiltonian function and the Riccati equation. The marginal Hamiltonian function will be an expression of the accounts based on optimal behaviour including both the change in result and the change in balance.

1. Introduction

Francesco Riccati was an Italian count who lived from 1676 to 1754. As well as enjoying a life of privilege he was also an enthusiastic disciple of mathematics. Riccati thereby became a significant contributor to the development of infinity calculations. In this connection he had an extended exchange of correspondence with the famous mathematicians G. W. Leibniz and J. Bernoulli. Riccati's name is particularly associated with a type of differential equation in which there are both linear and quadratic elements to the equation.

Today the Riccati equation emerges as the central solution equation in a control system for a linear process with an optimal criterion in quadratic form. The Riccati equation is often in the form of a matrix in that we normally have more than one state variable in our process

$$\dot{R}(t) = -R(t)A - A^T R(t) + R(t)BP^{-1}B^T R(t) - Q \tag{1}$$

When $R(t)$ is found the associated equations that together decide the optimal solution for our process are given by the contexts — the equations for 'the price vector' $\underline{p}(t)$

$$\underline{p}(t) = R(t)\underline{x}(t) \tag{2}$$

and the decision vector $\underline{u}(t)$

$$\underline{u}(t) = -P^{-1}B^T\underline{p}(t) \tag{3}$$

For our optimal criterion ω at time $t = 0$ we find

$$\omega(0) = \tfrac{1}{2}\underline{x}^T(0)R(0)\underline{x}(0) \tag{4}$$

The four equations above constitute the principal equations of the Riccati system in continuous-time form. As will be shown in section 5, there also exist equivalent equations in the discrete-time model (Balchen, 1978).

The main objective of this article is

- — to interpret the Riccati equation and the corresponding relations in terms of economics
- — to demonstrate that the equations of the Riccati system in discrete-time form are descriptive of marginal economic movement in time.

2. The marginal optimal system in economics

It is known from control theory that for small (marginal) deviations from the optimal paths an optimal control system can be described by a linear process description with an optimal criterion in quadratic form. In this connection we refer to (R. F. Stengel, 1994) where this is described in more detail both for a continuous-time and for a discrete-time model description.

In continuous-time form, we start with an object function which in this context we wish to minimise over the time period t_1 to t_2 (maximising ω entails a change of sign).

$$\omega = S(\underline{x}(t_2)) + \int_{t_1}^{t_2} \pi(\underline{x}, \underline{u}, t)dt \tag{5}$$

In equation (5) π is the current result, $\underline{x}$ the system's state vector and $\underline{u}$ the decision vector.

This expression is to be minimised having taken account of the process description of our system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \tag{6}$$

To find the optimal path $\hat{\underline{x}}(t)$ and $\hat{\underline{u}}(t)$ we apply the fundamental equations of the maximum principle. For the Hamiltonian we have

$$H = \pi(\underline{x}, \underline{u}, t) + \underline{p}^T \underline{f}(\underline{x}, \underline{u}, t) \tag{7}$$

where $\underline{p}$ is the impulse vector, which is named shadow price vector in economic theory. The Hamiltonian must have an absolute maximum (minimum) with respect to $\underline{u}$.

We further have that $\underline{p}(t)$ and $\underline{x}(t)$ are solutions to the equations

$$\dot{\underline{x}} = \frac{\partial H}{\partial \underline{p}}, \quad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{x}} \tag{8}$$

We have additional conditions at time $t = t_2$. In our examples the condition is that $\underline{x}(t_2)$ should be free, which implies that $\underline{p}(t_2) = 0$.

When in our further analysis we use the term system, it should be taken to mean that it is an economic system.

Around the optimal paths $\hat{\underline{x}}(t)$ and $\hat{\underline{u}}(t)$ our optimisation problem may be described by the following equations

$$\min \Delta\omega = \tfrac{1}{2}\Delta\underline{x}^T(t_2)S\Delta\underline{x}(t_2) + \frac{1}{2}\int_{t_1}^{t_2} (\Delta\underline{x}^T(\tau)Q(\tau)\Delta\underline{x}(\tau) + \Delta\underline{u}(\tau)^T P(\tau)\Delta\underline{u}(\tau) \tag{9}$$

$$+ \Delta\underline{u}^T(\tau)M^T(\tau)\Delta\underline{x}(\tau) + \Delta\underline{x}^T(\tau)M(\tau)\Delta\underline{u}(\tau))d\tau \tag{10}$$

for the system

$$\Delta\underline{\dot{x}} = A\Delta\underline{x} + B\Delta\underline{u} \tag{11}$$

$$\Delta\underline{x} = \underline{x} - \hat{\underline{x}},\ \Delta\underline{u} = \underline{u} - \hat{\underline{u}} \tag{12}$$

where

$$A(t) = \frac{\partial \underline{f}(\hat{\underline{x}}(t), \hat{\underline{u}}(t), t)}{\partial \hat{\underline{x}}(t)},\ B(t) = \frac{\partial \underline{f}(\hat{\underline{x}}(t), \hat{\underline{u}}(t), t)}{\partial \hat{\underline{u}}(t)} \tag{13}$$

The matrices P, Q, S and M are obtained from the second order derivatives of the Hamiltonian function $H(t)$ around the optimal trajectories, viz.

$$Q(t) = \frac{\partial^2 H}{\partial \underline{\hat{x}}^2(t)},\ P(t) = \frac{\partial^2 H}{\partial \underline{\hat{u}}^2(t)},\ S(t_2) = \frac{\partial^2 S}{\partial \underline{\hat{x}}^2(t_2)},\ M(t) = \frac{\partial}{\partial \underline{\hat{u}}(t)} \frac{\partial H}{\partial \underline{\hat{x}}(t)} \tag{14}$$

The solution for our optimisation problem is:

$$\Delta\underline{u} = -P^{-1}(M^T + B^T R)\Delta\underline{x} \tag{15}$$

where

$$\begin{aligned} \dot{R}(t) = {} & (-A^T + M^T P^{-1} B^T)R(t) + R(t)(-A + BP^{-1}M^T) \\ & + RBP^{-1}B^T R - Q + MP^{-1}M^T \end{aligned} \tag{16}$$

with the final condition $R(t_2) = S$. The equation above is a modified Riccati equation that can be translated to the desired form by setting

$$A' = A - BP^{-1}M^T,\ Q' = Q - MP^{-1}M^T \tag{17}$$

The optimal system described above, represented by a linear process description and quadratic elements in the criterion itself, is a first approximation of optimal economic behavior for marginal deviation around the paths $\hat{\underline{x}}(t)$ and $\hat{\underline{u}}(t)$. The deviation system ($\Delta\underline{x} = \underline{x} - \hat{\underline{x}}$, $\Delta\underline{u} = \underline{u} - \hat{\underline{u}}$) will, in an economic context, be the dynamic marginal optimal system. The paths $\underline{x}(t)$, $\underline{u}(t)$ represent the real system. Similarly the paths $\hat{\underline{x}}(t)$, $\hat{\underline{u}}(t)$ represent the reference system.

3. The marginal system — continuous time example

We have now come so far in our analysis that it is appropriate to introduce examples in economics to demonstrate the entirety. We will set the following condition to our main example:

> The example should have a form which enables the Riccati equation in continuous-time form to be solved analytically. This condition in itself demands that the model be simple.
>
> From this, we have come to the conclusion that a model where the process description is linear and the current result function is of the linear/quadratic form will be most suitable.

As will be shown, it will later be relatively simple to adapt the description to more complex mathematical problems.

3.1. *The model*

As a basis for the model, we will look at a 'firm' that is characterized by the following economic variables which will each be a function of time t.

Y = the firm's sales measured in the number of units sold u_s per time unit (u_s/t)
K = the firm's production capital measured in the number of production units (u_p)
J = investment in production capital calculated gross (u_p/t)

We have the following prices in our model

q_y = the price of the end product $(€/u_s)$
q = the price of capital equipment $(€/u_p)$
r = the rate of interest $(1/t)$
€ is the symbol for euro.

We assume that the firm's production (which we assume is the same as the firm's sale) can be described by a production function which has the following simple form

$$Y = a_1K - a_2K^2 \tag{18}$$

In the production function above a_1 and a_2 are positive constants. Normally a_1 will be significantly greater than a_2.

We now wish to introduce an element into our result function π that represents costs in connection with the decision to invest. Experience shows that installation of new equipment often is an activity that is additional to the firm's normal work and one which results in disproportionate costs in relation to the investment itself. In our interpretation context we shall use the following simple quadratic cost function

$$q_0J^2 \tag{19}$$

which will be a first approximation to costs such as overtime costs, installation costs and similar. The quadratic element should on average form a minor part of the actual value of the investment.

The elements portion of the value of the investment is given by the expression

$$\frac{q_0}{q} J \tag{20}$$

With the selected numerical figures in (23) the value will be less than 1%.

Our objective will now be that we wish to maximise accumulated cash flow over a given period of time T, which means that we in our example have no discount element included in our criterion.

Our object function ω (measured in €) takes the form

$$\omega = \int_0^T \pi dt = \int_0^T (q_y(a_1K - a_2K^2) - q_0J^2 - rqK)dt \tag{21}$$

Here the element $q_y(a_1K - a_2K^2)$ comprises the 'firm's revenue side', or more precisely the economic contribution before the cost of capital and costs related to the investment itself are taken into account, q_0J^2 is the quadratic element — and the element rqK comprises the capital cost of tying up capital in production. If we for simplicity reasons assume that labour is proportional with output, we can consider it to be hidden in the economic contribution.

In this model we have assumed that the capital equipment itself is leased. For this reason we have no element included in the current result π for the investment itself.

Our process equation is, assuming no depreciation

$$\dot{K} = J \tag{22}$$

In the calculations we shall assume the following set of figures in our parameters

$$r = 0.2,\ a_1 = 0.4,\ a_2 = 0.0909,\ q = q_y = 1,\ q_0 = 0.06,\ t_1 = 0,\ t_2 = T = 3 \tag{23}$$

The high value of r $(= 0.2)$ is due to the fact that the capital equipment is leased.

3.2. *Calculation of the optimal solution for the total system*

The maximum principle gives the following differential equation for the capital K, the optimal solution

$$2q_0 K'' - 2q_y a_2 K + (a_1 q_y - rq) = 0 \tag{24}$$

The capital K is constant for

$$K = \frac{a_1 q_y - rq}{2q_y a_2}, \tag{25}$$

As known, for our example the path given by (25) is characterized in the way that we have the marginal revenue of capital equal to the marginal expenditure i.e.

$$q_y \frac{\partial Y}{\partial K} = rq \tag{26}$$

Our system wants to move to this level. In the further calculations for the continuous-time case, we will be using equation (25) as a basis for the reference paths $\hat{K}$ and $\hat{J}$ in the marginal analysis.

We have

$$\hat{K} = \frac{a_1 q_y - rq}{2q_y a_2} = 1.1,\quad \hat{J} = 0 \tag{27}$$

We will change this assumption when we go over to a discrete-time model formulation.

The solution to (24) becomes

$$K(t) = \hat{K} + C_1 \exp(\beta t) + C_2 \exp(-\beta t) \tag{28}$$

$$\beta = \frac{1}{q_0}\sqrt{q_0 a_2 q_y} \tag{29}$$

$$p(t) = 2q_0 J(t) \tag{30}$$

where values of the constants C_1 and C_2 are deduced from the constraints $K(0) = 1.0$ and the shadow price at the end point in time $p(T)$ equal to 0. For the constants we find

$$C_1 = -6.1986 * 10^{-5},\ C_2 = -9.9938 * 10^{-2},\ \beta = 1.2309, \tag{31}$$

The curve for K will increase quickly from the start level $K(0) = 1$ to search the level $\hat{K} = 1.1$. At the end point in time we have $K(3) = 1.095$.

3.3. *Deviation between the actual system and the reference system*

The table below gives numerical values for key economic figures for the actual, reference and deviation systems at the time $t = 0$. At this point in time $\Delta K(0)$ is equal to -0.1. For the object function ω itself this is calculated over the whole interval in time i.e. from 0 to three years.

Results for the total system

		actual	**reference**	**deviation**	
shadow price	p	0.01475	0	0.01475	
investment	J	0.12293	0	0.12293	
current result	π	0.10819	0.11	-0.00181	(32)
price element	L_p	$1.8078 * 10^{-3}$	0	$1.8078 * 10^{-3}$	
Hamiltonian	H	≈ 0.11	0.11	$-2.25 * 10^{-6}$	
object function	ω	0.32926	0.33	$-7.376 * 10^{-4}$	

3.4. *The marginal system*

We will now compare the deviation results for the total system with what we find with our marginal system, which means that the Riccati equation must be solved. From the Hamiltonian function we find the expressions for Q and P (cf. (14)), we have $q_y = 1$

$$Q = \frac{\partial^2 H}{\partial \widehat{K}^2} = 2a_2, \ P = \frac{\partial^2 H}{\partial \widehat{J}^2} = 2q_0 \tag{33}$$

The change in sign is because we now minimise.

From the process equation (22) the 'matrices' A and B are found

$$A = 0, \ B = 1 \tag{34}$$

The Riccati equation in this context is

$$\dot{R} = R^2 \frac{1}{2q_0} - 2a_2 \tag{35}$$

where we have the constraint $R(3) = 0$. The equation itself is solvable, integration gives

$$R(t) = 2q_0 \sqrt{\frac{a_2}{q_0}} \frac{1 - e^{2\left(\sqrt{\frac{a_2}{q_0}}(t - T)\right)}}{1 + e^{2\left(\sqrt{\frac{a_2}{q_0}}t - T\right)}} \tag{36}$$

$R(t)$ will almost be constant in the first part of the period and then falls to 0 at the end. With the numerical value set selected we find the value at the initial point in time $R(0) = 0.14752$.

Above we have assumed that the time period $T = 3$. There is a borderline case when $T \to \infty$. In this event we can approximately set $\dot{R}(0) = 0$. We then find that $R(0) = 0.1477$. The deviation related to the value when $T = 3$ is in other words very small. (The cases when we can set $\dot{R}(0)$ equal to zero simplify the analytical solution considerably).

From the expression for $R(t)$ we can also find an expression for $\dot{R}(t)$. At the time $t = 0$ we find $\dot{R}(0)$ equal to -0.00045 — a negative but small figure. We can further deduce that $\dot{R}(t)$ will remain negative throughout the relevant interval and that the absolute value will increase when we approach the end point in time. This is a point to which we shall return from an interpretation viewpoint. With a start point in $R(0)$ we can now find the various key economic figures for the marginal system. The results are shown in the following table.

Comparisons of the marginal and the total system

	marginal system		**total system**	
	numerical result	equation	marginal dev.	
shadow price	-0.01475	$\Delta p = R\Delta x$	0.01475	
investment	0.12293	$\Delta u = -P^{-1}B^T\Delta \underline{p}$	0.12293	
current result	0.00181	$\Delta\pi = \frac{1}{2}(\Delta \underline{x}^T Q \Delta \underline{x} + \Delta \underline{u}^T P \Delta \underline{u})$	-0.00181	(37)
price element	$-1.8078 * 10^{-3}$	$\Delta L_p = \Delta \underline{p}^T \Delta \underline{\dot{x}}$	$1.8078 * 10^{-3}$	
Hamiltonian	$2.25 * 10^{-6}$	$\Delta H = \Delta\pi + \Delta L_p$	$-2.25 * 10^{-6}$	
object func.	$7.376 * 10^{-4}$	$\Delta\omega = \frac{1}{2}\Delta \underline{x}(0)^T R(0) \Delta \underline{x}(0)$	$-7.376 * 10^{-4}$	

As shown in the table the marginal system gives the same numerical answer as what we found from the total system even for a marginal deviation as great as $\Delta x(0) = \Delta K(0) = -0.1$.

We will develop the results of the table in more detail interpretationally in the following sections.

4. Economic interpretations related to the continuous-time model

4.1. *The optimal criterion*

As shown in table (37) the value of our optimal criterion in the marginal system, and where the objective has been to minimise, is given by the expression

$$\Delta\omega = \tfrac{1}{2}\Delta \underline{x}(0)^T R(0) \Delta \underline{x}(0) \qquad (38)$$

This equation says that our criterion at the instant of decision $t = 0$ is dependent only on the value of the marginal economic state vector $\Delta \underline{x}(0)$ (which in principle is given) and the Riccati matrix $R(0)$. Behind this recognition lies of course the fact that we have solved the Riccati differential equation over the period $(t_2, 0)$, having taken account of constraints given for the end point in time t_2. If we assume that $\Delta \underline{x}(0)$ represents a unit marginal increase it is the value of the individual parts of the matrix R that set the value of the criterion. We will therefore regard R as a matrix for marginal value stocks. Correspondingly the Riccati equation itself is an equation for marginal value streams to the 'stocks' R. In a non-regulated form, this value equation is given by, (see also Lystad (1974))

$$\dot{R} = -RA - A^T R - Q \qquad (39)$$

We can therefore consider R to be a state matrix in the marginal description. The element

$$RBP^{-1}B^T R \qquad (40)$$

in the value equation ensures that the solution is optimal.

It follows from the equation (38) that the matrix R allocates the criterion on the economic state variables $\Delta \underline{x}$ for the optimally controlled process (similarly the P and Q matrices in the expression of the running result). The matrix R therefore also represents weightings.

From table (37) we find the value of our object function in the marginal system.

$$\Delta\omega = 7.376*10^{-4} \tag{41}$$

In the same section we have calculated the economic result $\widehat{\omega}$ in the total system on the path $\widehat{K}$, $\widehat{J}$. Correspondingly we have calculated the result ω when we move on the actual path $K(t)$, $J(t)$. From this we get the following expression for the marginal result.

$$\omega - \widehat{\omega} = -7.376*10^{-4} \tag{42}$$

where the answer is the same as what the marginal system gives.

The expression for $\Delta\omega$ therefore represents a marginal economic loss in our system, a loss we have due to not being on the reference paths $\widehat{K}$, $\widehat{J}$.

4.2. *The Hamiltonian function — the marginal aggregated value stream*

It may now be of interest to analyse in more detail the marginal aggregated value flow, the Hamiltonian function for our system where the object function is to be minimised. At the point in time t (which is the current time) we have for the Hamiltonian function

$$\Delta H = \tfrac{1}{2}(\Delta \underline{u}^T P \Delta \underline{u} + \Delta \underline{x}^T Q \Delta \underline{x}) + \Delta \underline{p}^T (A \Delta \underline{x} + B \Delta \underline{u}) = \Delta\pi + \Delta L_p \tag{43}$$

If we now use the equations (2), (3) $\Delta\underline{p} = R\Delta\underline{x}$ and $\Delta\underline{u} = -P^{-1}B^T\Delta\underline{p}$ which are a part of the optimal solution we get

$$\Delta H = -\tfrac{1}{2}\Delta\underline{x}^T(-RA - A^TR + RBP^{-1}B^TR - Q)\Delta\underline{x} = -\tfrac{1}{2}\Delta\underline{x}^T\dot{R}\Delta\underline{x} \tag{44}$$

The equations above demonstrate clearly the close relationship there is between the Hamiltonian function and the Riccati equation itself.

The element

$$-\tfrac{1}{2}\Delta\underline{x}^T\dot{R}\Delta\underline{x} \tag{45}$$

is thus equal to the marginal value stream — the Hamiltonian function — for the optimal solution on the margin.

The table below shows how the different elements in the Riccati equation represent the elements $\Delta\pi$ and ΔL_p in the Hamiltonian function.

The Riccati equation and the Hamiltonian function

$$\begin{aligned}
\text{running result } \Delta\pi &= -\tfrac{1}{2}\Delta\underline{x}^T(-RBP^{-1}B^TR - Q)\Delta\underline{x} = -\tfrac{1}{2}\Delta\underline{x}^T(-\boldsymbol{\Delta\pi})\Delta\underline{x} \\
\text{price element } \Delta L_p &= -\tfrac{1}{2}\Delta\underline{x}^T(-RA - A^TR + 2RBP^{-1}B^TR)\Delta\underline{x} \\
&= -\tfrac{1}{2}\Delta\underline{x}^T(-\mathbf{\Delta L}_p)\Delta\underline{x} \\
\text{Hamiltonian } \Delta H &= -\tfrac{1}{2}\Delta\underline{x}^T\dot{R}\Delta\underline{x} = \Delta\pi + \Delta L_p
\end{aligned} \tag{46}$$

In the table above $\Delta\pi$ will be an expression of the current loss resulting from not being on the path $\widehat{K}$, $\widehat{J}$.

The table also illustrates the clear economic interpretation the different elements in the Riccati equation have. The Riccati matrix equation itself allocates the marginal aggregated value stream — the Hamiltonian function — on the economic state variables for the optimally controlled process. The matrix given by the expression

$$\boldsymbol{\Delta\pi} = RBP^{-1}B^TR + Q \tag{47}$$

therefore allocates the marginal result to the economic states. In the same way the matrix

$$-\mathbf{\Delta L}_p = -RA - A^TR + 2RBP^{-1}B^TR \tag{48}$$

allocates the price element to the economic states. We therefore can divide the Riccati equation in two

$$-\dot{R} = +\Delta\pi + \Delta L_p \tag{49}$$

Numerical values give (we have $n = 1$) $\Delta\pi(0) = 0.36315$, $\Delta L_p(0) = -0.3627$.

To make economic interpretations we will distinguish between

(a) the beginning of the optimisation interval and
(b) when we approach the end point in time.

For our chosen model with one state variable we have earlier shown that $R(t)$ is virtually constant in the beginning of the optimisation interval, which means that $\dot{R}$ is approximately equal to 0 but nevertheless negative. Being able to set $\dot{R}$ approximately equal to zero is a situation we will have when the time constants in the economic process are shorter than the duration of the optimisation period. $\dot{R}$ being approximately equal to zero also means that ΔH is approximately equal to zero, but positive in the marginal system. The economic interpretation of this is that at the instant of decision $t = 0$ we have approximately balance — but with a change of sign — between the running result element $\Delta\pi$ and the price element ΔL_p.

For the first part of the period we can set.

$$|\Delta\pi| \approx |\Delta L_p| \text{ and } |\Delta\pi| > |\Delta L_p| \tag{50}$$

Earlier we have pointed out that $R(t)$ falls when we approach the end point in time T, which means that the rate of change increases in absolute value with increasing time. In practice this means that the running result element $\Delta\pi$ has a significantly greater absolute value than the price element ΔL_p, which is quite natural from an economic viewpoint.

As known, the 'economic contribution result' may be used in two ways, either by reinvesting in the firm itself (represented by the price element) or by passing it as a return to the owners of the firm (the running result element).

When we approach the end point in time we will put no more effort in reinvesting in the firm, which explains why the price element approaches 0.

Development over time for $\Delta\pi$, ΔL_p, $\dot{R}$, ΔK, ΔH and R

t	$\Delta\pi$	ΔL_p	$\dot{R}$	ΔK	ΔH	R	
0	0.36315	−0.3627	$-4.5 * 10^{-4}$	−0.1	$2.25 * 10^{-6}$	0.14752	
1	0.35839	−0.35318	$-5.21 * 10^{-3}$	−0.0294	$2.25 * 10^{-6}$	0.14557	(51)
2	0.31095	−0.2583	$-5.27 * 10^{-2}$	−0.0092	$2.25 * 10^{-6}$	0.12449	
3	0.1818	0	−0.1818	−0.005	$2.25 * 10^{-6}$	0	

It follows from the table that ΔH is numerically constant over the optimisation interval. In the expression for ΔH ref. equation (46) the absolute value of ΔK is decreasing whilst $\dot{R}$ in absolute value increases with time.

4.3. *Main conclusions on the continuous-time model*

(1) Our chosen example is characterized by:

Linear process description, current result of the linear/quadratic form — as well as matrices P and Q being constant.

For this example, we find complete numerical agreement between the answers given by the Riccati system and what the total system gives (cf. table (37)).

(2) As we can see from (46), there is a close relationship between the marginal value stream, the Hamiltonian function ΔH for the optimal solution on the margin, and the Riccati equation itself

$$\Delta H = \Delta\pi + \Delta L_p = -\tfrac{1}{2}\Delta\underline{x}^T\dot{R}\Delta\underline{x} \tag{52}$$

where $\dot{R}$ represents the left side of the Riccati equation.

(3) With reference to equation (38)

$$\Delta\omega = \tfrac{1}{2}\Delta\underline{x}(0)^T R(0)\Delta\underline{x}(0) \tag{53}$$

$\Delta\omega$ represents a marginal economic loss in our system, a loss we have from not being on the reference paths $\widehat{K}$, $\widehat{J}$.

On the basis of the matrix R, higher weightings r_{ij} will mean that it has greater impact on the criterion. From an economics point of view, it is therefore more profitable to focus on the areas where r_{ij} has a high value.

5. The marginal system — discrete time model

5.1. *The principal equations of the Riccati system*

The description of the deviation between the real and the reference system in discrete-time form is given by the discrete-time equations of the Riccati system. These are briefly summarized below.

The discrete-time Riccati equation is

$$R(k) = Q + \Phi^T R(k+1)\Phi - \Phi^T R(k+1)\Gamma(P + \Gamma^T R(k+1)\Gamma)^{-1}\Gamma^T R(k+1)\Phi \tag{54}$$

or in simplified form

$$R(k) = Q + \Phi^T R(k+1)(I + \Gamma P^{-1}\Gamma^T R(k+1))^{-1}\Phi \tag{55}$$

Here Φ and Γ are matrices for the process description.

$$\Delta\underline{x}(k+1) = \Phi\Delta\underline{x}(k) + \Gamma\Delta\underline{u}(k) \tag{56}$$

The Q and P matrices are given by

$$Q(k) = \frac{\partial^2 H(k)}{\partial\underline{\widehat{x}}^2(k)},\quad P(k) = \frac{\partial^2 H(k)}{\partial\underline{\widehat{u}}^2(k)} \tag{57}$$

For the price vector and the decision vector, we have

$$\Delta\underline{p}(k+1) = \Phi^{-T}(R(k) - Q)\Delta\underline{x}(k) \tag{58}$$

$$\Delta\underline{u}(k) = -P^{-1}\Gamma^T\Phi^{-T}(R(k) - Q)\Delta\underline{x}(k) = -P^{-1}\Gamma^T\Delta\underline{p}(k+1) \tag{59}$$

where

$$\Delta\underline{u}(k) = u(k) - \underline{\widehat{u}}(k),\ \Delta\underline{x}(k) = \underline{x}(k) - \underline{\widehat{x}}(k) \tag{60}$$

The Hamiltonian function and the optimal criterion in the discrete-time form are given by the expressions, (see also Nyman (2004))

$$\Delta H(k) = (\Delta\underline{u}^T(k)P\Delta\underline{u}(k) + \Delta\underline{x}^T(k)Q\Delta\underline{x}(k)) + \Delta\underline{p}^T(k+1)\Delta\underline{x}(k+1) = \Delta\underline{x}^T(k)R(k)\Delta\underline{x}(k) \tag{61}$$

$$\Delta\omega(0) = \sum_{k=0}^{N-1}(\Delta\underline{u}^T(k)P\Delta\underline{u}(k) + \Delta\underline{x}^T(k)Q\Delta\underline{x}(k)) = \Delta\underline{x}^T(0)R(0)\Delta\underline{x}(0) \tag{62}$$

which gives

$$\Delta H(0) = \Delta\omega(0) = \Delta \underline{x}^T(0) R(0) \Delta \underline{x}(0) \tag{63}$$

5.2. *The discrete-time marginal system — a model for marginal movement in time*

So far, the Riccati system has been based on a comparison between the real and the reference systems at the same time *k*. The analysis is vertical and represented among others by the equations (59) and (60). We are now looking for a mathematical model which is valid horizontally, i.e. which describes the marginal optimal movement of the vectors $\underline{x}(k)$, $\underline{u}(k)$ from time k to the next time $k+1$. Marginal optimal movement in time must in this context be understood thus:

(1) The movement is in discrete steps and on a total level.
(2) Optimisation starts anew at each new step. This is congruent with normal economic behavior.
(3) The movement starts at time k = 0 from an initial level, i.e. a baselevel value in the total system. This baselevel value will normally be other than zero.
(4) The change from one level to the next does not necessarily have to be small (cf. the change of approximately 9% on $\underline{x}_1$ from time $k=1$ to $k=2$ in table (101)). However, the change will normally be small compared to the baselevel value. In the example, the baselevel value is $x_1(0) = 1$.

To find a horizontal model, we will split the problem in two — first we find the mathematical model which is valid under constant economic constraints, then we expand the model to handle changes in the constraints.

5.2.1. *Constant economic constraints.* In the continuous-time example, we chose the paths given by equation (27) as reference paths. We are now going to alter this. We will make use of the fact that values of the state vector and the decision vector in the general form are governed by the process description

$$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), k) \tag{64}$$

and the optimal criterion

$$\omega = S(\underline{x}(N)) + \sum_{k=0}^{N-1} \pi(\underline{x}(k), \underline{u}(k), k) \tag{65}$$

Values of the state vector and the decision vector being based on optimal behavior, these can be used as reference in the marginal analysis. For the state vector $\hat{\underline{x}}(k)$ in the reference system at time k, we therefore choose the value which appears in the real system one period into the future. We get

$$\hat{\underline{x}}(k) = \underline{x}(k+1) \tag{66}$$

Similarly, a natural estimate of $\hat{\underline{u}}(k)$ would be the optimal decision vector at time $k+1$ based on the initial value $\underline{x}(k+1)$, and where we optimize over the interval from k + 1 and *N* periods ahead.

Given $\hat{\underline{x}}(k) = \underline{x}(k+1)$, for the case when the economic situation on which we base ourselves for optimisation is identical whether one starts at time k and goes N periods into the future, or makes a new start from time $k+1$, the two calculations will give the same result. We get

$$\hat{\underline{u}}(k) = u'(k+1) \tag{67}$$

(The index ′ makes reference to the new start from time k + 1). For the deviations, we now have

$$\Lambda \underline{x}(k) = \underline{x}(k) - \underline{x}(k+1),\ \Lambda \underline{u}(k) = \underline{u}(k) - \underline{u}'(k+1) \tag{68}$$

(the symbol Λ is used when there is a marginal change in time as defined in (68))

Equation (68) is horizontal and replaces equation (60). From equation (59) we find

$$\underline{u}(k) - \hat{\underline{u}}(k) = -P^{-1}\Gamma^T\Phi^{-T}(R(k) - Q)(\underline{x}(k) - \hat{\underline{x}}(k)) \tag{69}$$

We now use the relations

$$\hat{\underline{u}}(k) = \underline{u}'(k+1),\ \hat{\underline{x}}(k) = \underline{x}(k+1) \tag{70}$$

which give the decision vector $\underline{u}'(k+1)$ with a new start at time $k+1$.

$$\underline{u}'(k+1) = P^{-1}\Gamma^T\Phi^{-T}(R(k) - Q)\Lambda \underline{x}(k) + \underline{u}(k) \tag{71}$$

NUMERICAL EXAMPLE The main structure of the discrete-time model corresponds to the continuous-time model as it is presented in section 3. We now, in addition, wish to obtain a model where there is the closest possible agreement numerically between the answers given by the maximum principle and those we get from the Riccati equations. The process equation in discrete-time form is

$$K(k+1) = K(k) + J(k) \tag{72}$$

The discrete-time optimal criterion becomes analogous to equation (21)

$$\omega = \sum_{k=0}^{2} \pi(k) = \sum_{k=0}^{2} (a_1 K(k) - a_2 K(k)^2 - q_0 J(k)^2 - rqK(k)) \tag{73}$$

We will use the same numerical values as in the continuous-time case. The maximum principle gives the following optimal values for investment and capital with the starting times $k=0$ and $k=1$ ($k'=0$).

The real system — optimal values

	k	0	1	2	3	4
J(k)		**0.068**	*0.0193*	*0*		
K(k)		**1**	*1.068*	*1.0873*	*1.0873*	
		k′	**0**	**1**	**2**	**3**
J′(k′)			**0.0219**	*0.0062*	*0*	
K(k′)			**1.068**	*1.0899*	*1.0961*	*1.0961*

(74)

For the marginal system, we find

$$P = 0.12,\ Q = 0.1818,\ \Gamma = \Phi = 1 \tag{75}$$

which gives $R(0) = 0.2633$. Placed in equation (71), this gives

$$\underline{u}'(1) = P^{-1}\Gamma^T\Phi^{-T}(R(0) - Q)(\underline{x}(0) - \underline{x}(1)) + \underline{u}(0) = 0.0219 \tag{76}$$

which is the same as we obtain from (74) ($J'(1) = 0.0219$). We will now use equation (76) together with the process equation (72) to calculate marginal optimal movement in time. $K(0)$, $J(0)$ are the initial conditions to get started with the Riccati model. This pair of figures also raises the model to a total level.

Marginal optimal movement in time — total values

	true value		Ric.system			
k	K	J	K	J		
0	1	0.068	*1.000*	*0.068*	initial cond. Ric.	(77)
1	1.068	0.0219	1.068	0.0219		
2	1.0899	0.007	1.0898	0.007		
3	1.0969	0.0022	1.0969	0.0022		

The calculation which shows the total values for K and J is effected with the help of the maximum principle (true value) as well as the Riccati system. The values of K and J for the first two years correspond to (74).

As we can see from table (77) when the initial conditions are given in the Riccati model, there is for our chosen example full numerical agreement between the two methods to describe marginal optimal movement in time.

Table (80) gives the important economic variables for the marginal system with starting time $k = 0$. The reference system will be the lowest values for $J'(k')$, $K(k')$ in table (74), displaced by one period to the left.

For the shadow price $\Delta p(1)$, the table gives us (see also (Kreyberg, 1969))

$$\Delta p(1) = \frac{\sum_{k=1}^{k=2} \Delta\pi(k)}{\Delta x(1)} = -5.56 \times 10^{-3} \tag{78}$$

which gives

$$\Delta L_p(0) = \Delta p(1) * \Delta x(1) = \sum_{k=1}^{k=2} \Delta\pi(k) = 1.22 \times 10^{-4} \tag{79}$$

Economic variables for the marginal system

k	0	1	2	3	
Δu(k)	.0461	0.013	0		
R(k)	0.2633	0.2541	0.1818	0	
Δp(k)	-1.79×10^{-2}	-5.56×10^{-3}	-1.6×10^{-3}	0	
Δx(k)	-0.068	-0.0219	-0.0088	-0.0088	(80)
ΔL_p(k)	1.22×10^{-4}	1.4×10^{-5}	0		
$\Delta\pi$(k)	1.07×10^{-3}	1.08×10^{-4}	1.4×10^{-5}		
ΔH(k)	1.2×10^{-3}	1.219×10^{-4}	1.4×10^{-5}		
$\Delta\omega$(0)	1.2×10^{-3}				

We will return to this in section 6 when we analyse in greater detail the economic result development.

5.2.2. *Change in the economic constraints.* The premises for the equations above are as previously mentioned that the 'economic constraints' which one bases oneself upon for optimisation are identical whether one starts at time k, or makes a new start at time $k+1$. We now wish to expand our model to handle changes in the economic constraints.

We define an expanded system with a new state vector which includes both our process and the neighborhood.

$$\Delta\underline{x} = \begin{bmatrix} \Delta\underline{x}_1 \\ \Delta\underline{x}_2 \end{bmatrix} \tag{81}$$

Here $\Delta\underline{x}_1$ represents our process, and $\Delta\underline{x}_2$ the neighborhood.

We hereby refer to (Balchen, 1978) where the theory is described in greater detail under the condition that the neighborhood influences the process only. In our context, we will assume that the change in the economic constraints can influence both the process and the optimal criterion.

As we are, for the neighborhood, looking for a model of change in the external constraints from one period to the next, the mathematical formulation will be of the form

$$\Delta\underline{x}_2(k+1) = \Phi_{22}\Delta\underline{x}_2(k) \tag{82}$$

The expanded system is now described by the equation

$$\Delta\underline{x}(k+1) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix} \Delta\underline{x}(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \underline{u}(k) \tag{83}$$

which in compact form becomes

$$\Delta\underline{x}(k+1) = \tilde{\Phi}\Delta\underline{x}(k) + \tilde{\Gamma}\Delta\underline{u}(k) \tag{84}$$

For the marginal description, the object function is given by equation (62)

$$\omega = \sum_{k=0}^{N-1} (\Delta\underline{u}^T(k)P\Delta\underline{u}(k) + \Delta\underline{x}^T(k)\tilde{Q}\Delta\underline{x}(k)) \tag{85}$$

The matrix $\tilde{Q}$ can be divided into four submatrices

$$\tilde{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{86}$$

Q_{11} will normally be other than zero.

As mentioned above, the change in economic constraints can also influence our optimal criterion. This can happen in the way that some of the remaining submatrices may also be other than zero.

Equation (85) ensures that the economics optimal criterion given by (65) is drawn into the analysis. The matrices P and $\tilde{Q}$ are found from the equation (57).

The solution to this problem will be in the form (cf. equation (59))

$$\Delta\underline{u}(k) = -P^{-1}\tilde{\Gamma}^T\tilde{\Phi}^{-T}(R(k) - \tilde{Q})\Delta\underline{x}(k) \tag{87}$$

where $R(k)$ is the solution of the Riccati equation.

$$R(k) = \tilde{Q} + \tilde{\Phi}^T R(k+1)(I + \tilde{\Gamma}P^{-1}\tilde{\Gamma}^T R(k+1))^{-1}\tilde{\Phi} \tag{88}$$

The matrix R may also be divided into four submatrices

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{89}$$

The optimal control variable with constant constraints is given by equation (71)

$$\underline{u}_1'(k+1) = P^{-1}\Gamma^T\Phi_{11}^{-T}(R_{11}(k) - Q_{11})(\underline{x}_1(k) - \underline{x}_1(k+1)) + \underline{u}_1(k) \tag{90}$$

The 'additional control variable' $\Delta\underline{u}_2'$ which one gets as a result of altered constraints is given by the expression

$$\Delta\underline{u}_2'(k+1) = P^{-1}\Gamma^T R_{12}(k+1)(I+\Gamma P^{-1}\Gamma^T R_{11}(k+1))^{-1}\Phi_{22}\Delta\underline{x}_2(k) \tag{91}$$

The control variable $u'(k+1)$ which gives a new start at time $k+1$ is therefore equal to

$$u'(k+1) = u_1'(k+1) + \Delta\underline{u}_2'(k+1) \tag{92}$$

NUMERICAL EXAMPLE For the neighborhood, we assume that the change in constraints is due to technological changes. We therefore start by assuming that the technological level comes in multiplicatively in the production function with the expression

$$x_2(k) = 1.0\mu^k \tag{93}$$

The process equation related to the neighborhood thus becomes

$$x_2(k+1) = 1.0\mu * x_2 \tag{94}$$

which for the change description gives, cf. equation (82)

$$\Phi_{22} = 1.0\mu \tag{95}$$

In our example, we set $\mu = 9$ i.e. a growth of 9% per annum (we have assumed such a high figure in order to show the equal suitability of the model for high rates of growth). Our process equation remains unchanged ($x_1 = K$, $u = J$)

$$x_1(k+1) = x_1(k) + u(k) \tag{96}$$

The current result function is changed to

$$\pi(k) = x_2(k)(a_1x_1(k) - a_2x_1(k)^2) - q_0u(k)^2 - rqx_1(k) \tag{97}$$

In this example, optimalisation is effected over two years.

Further, we have for the marginal model

$$\Gamma = 1, \; P = 0.12 \tag{98}$$

The $\tilde{Q}$ matrix is given by (57). In the matrix we have that $q_{12}(k)$ is given by the expression

$$q_{12}(k) = -a_1 + 2a_2\hat{x}_1(k) \tag{99}$$

We now make use of (66) which gives

$$q_{12}(k) = -a_1 + 2a_2x_1(k+1) \tag{100}$$

implying that we must know $u'(1)$ in addition to $x_1(0)$, $x_2(0)$ and $u(0)$ to be able to get started with the marginal model. We assume $x_1(0) = x_2(0) = 1.0$. With these assumptions, the table below gives marginal movement in time when the constraints change.

Table (101) shows that now too there is numerical agreement between the answers given by the maximum principle and what we find using the Riccati equations.

The total Riccati equation given in (88) will be the principal solution equation.

Together with the control variable equations (90), (91), (92) and the process equation (84), these will be the fundamental equations which describe marginal movement in time.

The solution is in two parts in the sense that we have a feedback solution from our economic main process assuming constant constraints, and a 'feedforward solution' from the neighborhood by $\Delta \underline{u}_2'$, and which ensures that the constraints are altered.

In the chosen example with 9% external growth per annum, $\Delta \underline{u}_2'$ is larger than $\underline{u}_1'$. When the economic growth rate is more normal, for example 2 to 3% per annum, $\underline{u}_1'$ will be by far the largest.

Marginal optimal movement in time, two states — total values

k	x_1	x_2	$u' = u_1' + \Delta u_2'$	u_1'	$\Delta u_2'$		
0	1	1	.1189				
1	1.1189	1.09	.0999	.0449	.0550	true value	
			.0999	.0449	.0550	Ric.system	
2	1.2188	1.09^2	.0874	.0357	.0517	true value	(101)
			.0874	.0357	.0517	Ric.system	
3	1.3062	1.09^3	.0781	.0295	.0486	true value	
			.0781	.0295	.0486	Ric.system	

5.2.3. *The equations of the marginal system applied to a Cobb-Douglas production function.* So far, we have assumed that the $Q(k)$ 'matrix' has been constant. We will now alter this by replacing equation (18) with the well known Cobb-Douglas production function of the form

$$Y = \vartheta * K^{\alpha} \tag{102}$$

For the two constants in the equation we choose the following numerical values

$$\vartheta = 0.3463, \quad \alpha = 0.6 \tag{103}$$

The remaining numerical values are identical to those in our main example. Optimalisation is effected over three years.

The $Q(k)$ 'matrix' is now given by the expression

$$Q(k) = \vartheta * \widehat{K}(k)^{\alpha - 2} * \alpha(1 - \alpha) \tag{104}$$

which implies that $Q(k)$ varies.

With this as a starting point, we wish to effect a comparison of the results between the two calculation methods.

Marginal optimal movement in time, Cobb-Douglas production function — total values

k	$K(k)_{true}$	$J(k)_{true}$	$K(k)_{ric}$	$J'(k)_{ric}$	
0	1	0.0503	*1*	*0.0503*	
1	1.0503	0.0248	1.0503	0.0247	(105)
2	1.0751	0.0125	1.0750	0.0123	
3	1.0876	0.0063	1.0873	0.0061	

When $Q(k)$ varies, we obtain a very small numerical deviation between the two methods when we calculate K and J.

5.3. *Main conclusions on the discrete-time model*

(1) Even for our growth model described in section 5.2.2 where technological development comes in multiplicatively in the production function and where we have two states, there is full numerical agreement between the answers given by the maximum principle and those we get from the Riccati equation.

This conclusion is also valid for high growth rates.

(2) For the Cobb-Douglas example described in section 5.2.3, we get a numerical deviation between the two methods.

However, it must be recognized that the deviation is very small. This is in part due to the fact that the working point $\widehat{K}(k)$ in equation (104) is 'corrected' by the model itself when we move forward in time.

The calculation in the Cobb-Douglas example is effected for a period of time of one year (the time from k to $k+1$). Provided that the length of the period is reduced, the numerical deviation will be further reduced.

(3) The examples above demonstrate the article's main objective which is to show that the Riccati equation describes marginal economic movement in time.

(4) With our chosen examples, we have so far almost had full numerical agreement between the answers given by the maximum principle and those we obtain from the Riccati equation. To analyse the deviation problems in detail will constitute a complete research project in itself.

6. Analysis of the economic changes in the accounts

6.1. *The general equations*

With reference to point 5.1, we have for $\Delta H(k)$

$$\Delta H(k) = \Delta\underline{x}(k)^T R(k) \Delta\underline{x}(k) = \Delta\pi(k) + \Delta L_p(k) \tag{106}$$

Equation (106) shows that there is also in the discrete-time case a close relationship between the Hamiltonian function and the Riccati equation itself, in that $R(k)$ represents the Riccati equation's left side.

For the result $\Delta\pi(k)$ we find

$$\Delta\pi(k) = \Delta\underline{x}^T(k)(Q + \Phi^T Z(k+1)^{-T} R(k+1)\Gamma P^{-1}\Gamma^T R(k+1) Z(k+1)^{-1}\Phi)\Delta\underline{x}(k) \tag{107}$$

where

$$Z(k+1) = (I + \Gamma P^{-1}\Gamma^T R(k+1)) \tag{108}$$

Similarly, for the price element we find

$$\Delta L_p(k) = \Delta\underline{x}^T(k)(R(k) - Q - \Phi^T Z(k+1)^{-T} R(k+1)\Gamma P^{-1}\Gamma^T R(k+1) Z(k+1)^{-1}\Phi)\Delta\underline{x}(k) \tag{109}$$

With reference to equation (106), the Riccati matrix equation allocates the Hamiltonian function on the economic state variables for the optimally controlled process in a manner which is totally equivalent to the continuous-time case.

Further, the Riccati equation can also in the discrete-time case be split into two parts

$$R(k) = \boldsymbol{\Delta\pi}(k) + \boldsymbol{\Delta L}_p(k) \tag{110}$$

where $\boldsymbol{\Delta\pi}(k)$ is a 'profit matrix' and $\boldsymbol{\Delta L}_p(k)$ is the 'price element' matrix.

The matrix given by the expression

$$\boldsymbol{\Delta\pi}(k) = Q + \Phi^T Z^{-T} R(k+1)\Gamma P^{-1}\Gamma^T R(k+1) Z^{-1}\Phi \tag{111}$$

allocates the marginal result $\Delta\pi(k)$ to the economic states. In the same way the matrix

$$\mathbf{\Delta L}_p(k) = R(k) - Q - \Phi^T Z^{-T} R(k+1)\Gamma P^{-1}\Gamma^T R(k+1) Z^{-1}\Phi \tag{112}$$

allocates the price element $\Delta L_p(k)$ to the economic states.

6.2. *Numerical example from time $k=0$ to $k=1$*

The example is the same as that which is presented in section 5.2.1. Table (80) gives the principal economic variables for the marginal system.

6.2.1. *Changes in the running result.* From equation (107) we find the expression for $\Lambda\pi(0)$

$$\Lambda\pi(0) = \Lambda\underline{x}^T(0)(Q + \Phi^T Z^{-T}(1) R(1)\Gamma P^{-1}\Gamma^T R(1) Z^{-1}(1)\Phi)\Lambda\underline{x}(0) \tag{113}$$

The numerical values give

$$\Lambda\pi(0) = 1.07 \times 10^{-3} \tag{114}$$

Similarly, we obtain from the total system

$$\pi(0) = .10882, \ \pi'(1) = .10989 \tag{115}$$

which gives the same answer for the difference.

$\Lambda\pi(0)$ gives an expression of the total system's change in running result over the period $k=0$ to $k=1$.

6.2.2. *The change in balance.* From equation (109) we find the expression for $\Lambda L_p(0)$

$$\Lambda L_p(0) = \Lambda\underline{x}^T(0)(R(0) - Q - \Phi^T Z^{-T}(1) R(1)\Gamma P^{-1}\Gamma^T R(1) Z^{-1}(1)\Phi)\Lambda\underline{x}(0) \tag{116}$$

The numerical values give

$$\Lambda L_p(0) = 1.22 \times 10^{-4} \tag{117}$$

$\Lambda L_p(0)$ gives an expression for the total system's change in balance over the period $k=0$ to $k=1$, represented through future marginal result improvements, cf. equation (79).

$$\sum_{k=1}^{k=2} \Delta\pi(k) \tag{118}$$

In a general manner

$$\Lambda L_p(0) = \sum_{k=1}^{N-1} \Delta\pi(k) \tag{119}$$

by having for the reference system

$$\hat{\underline{x}}(0) = \underline{x}(1) \tag{120}$$

7. Main conclusions

The Riccati equation is an economic fundamental equation. There are two main reasons for drawing this conclusion.

(1) The equations of the Riccati system are marginally descriptive in time for the large majority of economic systems, i.e. those which can fit in with the principal

equations and inequalities which form the basis of the marginal model. The starting point is the general equations for the total system — the process description of the discrete-time form

$$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), k) \tag{121}$$

and the optimal criterion

$$\omega = S(\underline{x}(N)) + \sum_{k=0}^{N-1} \pi(\underline{x}(k), \underline{u}(k), k) \tag{122}$$

By knowing the fundamental equations in the marginal movement, we can get a broader insight in the marginal states and parameters. This is difficult to achieve with the total system as the relations are normally unknown.

(2) There is a close relationship between the marginal Hamiltonian function $\Lambda H(k)$ and the Riccati equation itself given by the relation (cf. (106))

$$\Lambda H(k) = \Lambda\underline{x}(k)^T R(k) \Lambda\underline{x}(k) = \Lambda\pi(k) + \Lambda L_p(k) \tag{123}$$

$\Lambda H(k)$ will be an expression of the accounts based on optimal behavior.

We have

- — $\Lambda\pi(k)$ is the change in result which one obtains over the period k to $k+1$ for the *total* system.
- — $\Lambda L_p(k)$ will be an expression of the *total system's* change in balance over the same period.

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