Exponential Stabilization of an Underactuated Surface Vessel

K. Y. PETTERSEN and O. EGELAND

The paper shows that a large class of underactuated vehicles cannot be asymptotically stabilized by either continuous or discontinuous state feedback. Furthermore, stabilization of an underactuated surface vessel is considered. Controllability properties of the surface vessels is presented, and a continuous periodic time-varying feedback law is proposed. It is shown that this feedback law exponentially stabilizes the surface vessel to the origin, and this is illustrated by simulations.

1. Introduction

Control of underactuated vehicles, i.e. vehicles where the control vector has lower dimension than the configuration vector, is a field of increasing interest. It has been studied by e.g. Byrnes and Isidori (1991) who gave results on stabilizability of a class of underactuated vehicles. Leonard (1995) shows how open loop small-amplitude periodic time-varying forcing can be used to control both underactuated spacecraft and underwater vehicles. Morin, Samson, Pomet and Jiang (1995) present smooth time-varying feedback laws that asymptotically stabilize an underactuated spacecraft. Both M'Closkey and Murray (1993) and Morin and Samson (1995) present important tools for analysis and exponential stabilization of underactuated vehicles.

Byrnes and Isidori (1991) show that underactuated vehicles with zero gravitational and buoyant field cannot be asymptotically stabilized by C^1 state feedback. We here extend their result by considering underactuated vehicles with a more general gravitational and buoyant field. We show that the vehicles cannot be asymptotically stabilized by either continuous or discontinuous state feedback.

Dynamic positioning of surface vessels is required e.g. in many offshore oil field operations such as drilling, pipe-laying and diving support. Critical to the success of a dynamically positioned surface vessel is its capability for accurate and reliable control, subject to environmental disturbances as well as to configuration related changes, such as a reduced number of available control inputs. This reduction may be the result of an actuator failure or a deliberate decision to limit the number of actuators due to e.g. cost and weight considerations.

In this paper we consider control of a surface vessel without side thruster. The problem considered is to find a control law that asymptotically stabilizes the surface vessel to the origin. As open loop control does not compensate for disturbances and

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Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7034, Trondheim, Norway. Phone: +47 73 59 43 76. Fax: +47 73 59 43 99. Email: KristinY.Pettersen@itk.ntnu.no Olav.Egeland@itk.ntnu.no
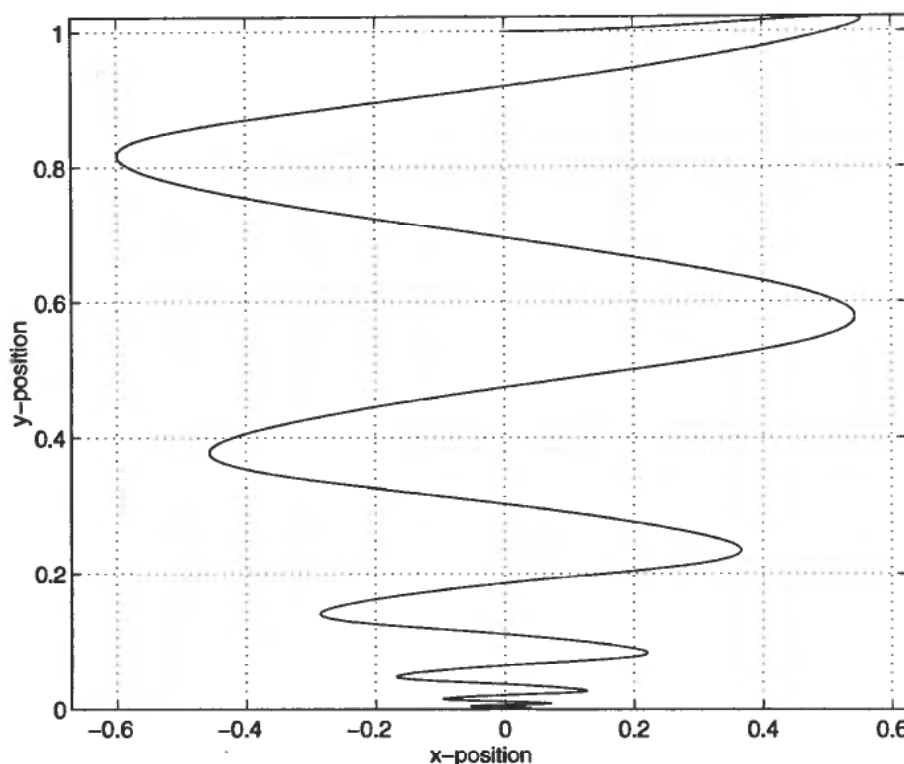
Figure 1.   Motion of the surface vessel.

model errors, we seek a feedback law. A question to be answered is whether the surface can be asymptotically stabilized with only two controls.

The stabilizability result gives that the underactuated surface vessel cannot be asymptotically stabilized by either continuous or discontinuous state feedback. We show however that the surface vessel still is locally strongly accessible and small time locally controllable. Furthermore we show that by introducing explicit time-dependence in the controller, the surface vessel can be asymptotically stabilized.

However it is not trivial to find such a feedback law. One approach may be the use of the analysis tools of M'Closkey and Murray (1993) and Morin and Samson (1995). For these tools to apply however, the system has to consist of homogeneous vector fields. The surface vessel model includes trigonometric terms, and is thus not homogeneous with respect to any dilation. To overcome this problem, we propose a global coordinate transformation that renders the system homogeneous.

Furthermore we propose a continuous periodic time-varying feedback law. We show that it asymptotically stabilizes, and in fact yields exponential convergence for the underactuated surface vessel to the origin. This is illustrated by simulations.

The paper is organized as follows. Stabilizability of a class of underactuated vehicles is addressed in Section 2. In Section 3 controllability properties of a surface vessel with only two controls are given. In Section 4 a coordinate transformation is proposed that renders the surface vessel model homogeneous, and a continuous periodic time-varying feedback law is proposed that exponentially stabilizes the surface vessel to the origin. This is illustrated by simulations in Section 5.

2. Stabilizability of underactuated vehicles

We consider underactuated vehicles of the form

$$M\dot{v} + C(v)v + D(v)v + g(\eta) = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{1}$$

$$\dot{\eta} = J(\eta)v \tag{2}$$

where $\eta \in \mathbb{R}^{n_1}, v \in \mathbb{R}^{n_2}, n_1 \geq n_2, \tau \in \mathbb{R}^m, m < n_2$. The matrix M is nonsingular and $\dot{M} = 0$. The matrix J has full rank ($\text{rank}(J) = n_2$).

This class includes e.g. underactuated surface vessels, underwater vehicles and spacecraft. The vector v denotes the linear and angular velocities with coordinates in the body-fixed frame, η denotes the position and orientation with coordinates in the earth-fixed frame, and τ denotes the control forces and torques with coordinates in the body-fixed frame. M is the inertia matrix, including added mass. $C(v)$ is the Coriolis and centripetal matrix, also including added mass. $D(v)$ is the damping matrix and $g(\eta)$ is the vector of gravitational and buoyant forces and torques. Equation (2) represents the kinematics.

The following proposition show that the gravitational and buoyant vector, $g(\eta)$, is important for the stabilizability properties of underactuated vehicles. The vector $g(\eta)$ consists of elements corresponding to the actuated dynamics, $g^a(\eta)$, and elements corresponding to the unactuated dynamics, $g^u(\eta)$:

$$g(\eta) = \begin{bmatrix} g^a(\eta) \\ g^u(\eta) \end{bmatrix} \tag{3}$$

$$g^a(\eta): \mathbb{R}^{n_1} \to \mathbb{R}^m \quad g^u(\eta): \mathbb{R}^{n_1} \to \mathbb{R}^{n_2 - m}$$

Proposition 1   *Consider the system (1)–(2), and suppose that $(\eta, v) = (0,0)$ is an equilibrium of the system. If $g^u(\eta)$ has a zero element then there exists no continuous or discontinuous state feedback law, $\alpha(\eta, v): \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$, that makes $(0,0)$ asymptotically stable.*

Proof.   Consider the mapping $f(\eta, v, \tau): \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1 + n_2}$ defined by

$$f(\eta, v, \tau) = \begin{bmatrix} J(\eta)v \\ -M^{-1}(C(v)v + D(v)v + g(\eta) - \begin{bmatrix} \tau \\ 0 \end{bmatrix}) \end{bmatrix} \tag{4}$$

Without loss of generality, we assume that

$$g^u(\eta) = \begin{bmatrix} g_1^u(\eta) \\ 0 \end{bmatrix} g_1^u(\eta): \mathbb{R}^{n_1} \to \mathbb{R}^{n_2 - m - 1} \tag{5}$$

Consider points of the form

$$\varepsilon = \begin{bmatrix} 0_{n_1 \times 1} \\ M^{-1} \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \beta \end{bmatrix} \end{bmatrix} \tag{6}$$

where $\alpha^1 \in \mathbb{R}^m, \alpha^2 \in \mathbb{R}^{n_2 - m - 1}$ are arbitrary vectors, and $\beta \in \mathbb{R}$ is an arbitrary non-zero number. As $J(\eta)$ has full rank, the equation $f(\eta, v, \tau) = \varepsilon$ implies $v = 0$. Thus it implies
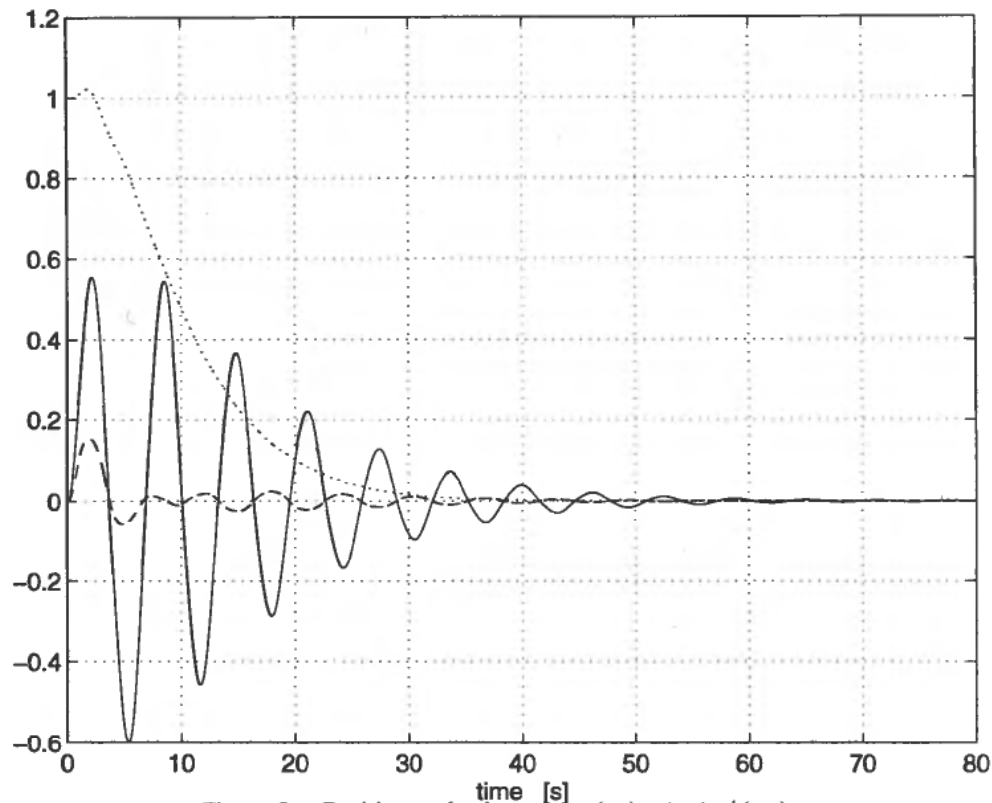
Figure 2.   Position and orientation $x(\text{---})$, $y(\cdots)$, $\psi(\text{--})$.

$$\begin{bmatrix} \tau - g^a(\eta) \\ - g_1^u(\eta) \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \beta \end{bmatrix} \qquad (7)$$

which has no solution (η, v, τ) as β is non-zero. As M is constant and nonsingular, and $\alpha^1, \alpha^2, \beta$ are arbitrary vectors, points of the form ε are contained in every neighborhood of zero in $\mathbb{R}^{n_1 + n_2}$. Thus the mapping f is not onto any neighborhood of 0 in $\mathbb{R}^{n_1 + n_2}$. This implies that there does not exist any continuous state feedback $\alpha(\eta, v): \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ such that $(\eta, v) = (0, 0)$ is an asymptotically stable equilibrium (Zabczyk 1989). As the system (1)–(2) is affine and cannot be asymptotically stabilized by continuous state feedback, it cannot be asymptotically stabilized by discontinuous state feedback either (Coron and Rosier 1994).   □

3.   Controllability properties of an underactuated surface vessel

We consider the surface vessel described by the following model:

$$M\dot{v} + C(v)v + Dv = \tau \qquad (8)$$

$$\dot{\eta} = J(\eta)\gamma \qquad (9)$$

where

$$C(v) = \begin{bmatrix} 0 & 0 & -m_{22}v \\ 0 & 0 & m_{11}u \\ m_{22}v & -m_{11}u & 0 \end{bmatrix} \qquad (10)$$

$$J(\eta) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$M = \text{diag}\{m_{11}, m_{22}, m_{33}\}$, $D = \text{diag}\{d_{11}, d_{22}, d_{33}\}$.

The vector $v = [u, v, r]^T$ denotes linear velocities in surge and sway, and angular velocity in yaw. The vector $\eta = [x, y, \psi]^T$ denotes position and orientation in earth-fixed coordinates. The vector $\tau = [\tau_1, \tau_2, \tau_3]^T$ denotes the control forces in surge and sway, and control torque in yaw. The inertia matrix M and the damping matrix D are constant and positive definite.

We consider the case where the surface vessel has no side thruster, i.e. $\tau_2 = 0$, because this is a common thruster configuration. However, the controllability analysis and control synthesis of this paper is easily extended to the cases where τ_1 or τ_3 are missing.

From Proposition 1 we know that the surface vessel cannot be asymptotically stabilized by either continuous or discontinuous state feedback, if any of the controls are missing. However we show here that the surface vessel yet is locally strongly accessible and small time locally controllable. Furthermore we show that by introducing explicit time dependence in the controller, asymptotic stability may be obtained.

Let

$$f(\eta, v) = \begin{bmatrix} J(\eta)v \\ -M^{-1}(C(v) + D)v \end{bmatrix} \tag{12}$$

$$g_1 = [0 \quad 0 \quad 0 \quad \tfrac{1}{m_{11}} \quad 0 \quad 0]^T \qquad g_2 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{m_{33}}]^T \tag{13}$$

Then the system (8)–(9) with $\tau_2 = 0$ may be written

$$\begin{bmatrix} \dot{\eta} \\ \dot{v} \end{bmatrix} = f(\eta, v) + g_1\tau_1 + g_2\tau_3 \tag{14}$$

Lemma 1 *The underactuated surface vessel (14) is locally strongly accessible* $\forall (\eta, v) \in \mathbb{R}^6$.

Proof. The vector fields $g_1, g_2, [f, g_1], [f, g_2], [g_2, [f, g_1]], [g_2, [[g_1, f], f]]$ span a six-dimensional space at every $(\eta, v) \in \mathbb{R}^6$. Thus the strong accessibility rank condition is satisfied, and consequently (14) is locally strongly accessible $\forall (\eta, v) \in \mathbb{R}^6$ (Nijmeijer and van der Schaft 1990). □

Proposition 2 *The underactuated surface vessel (14) is small time locally controllable from any equilibrium.*

Proof. From Lemma 1 we know that the system (14) is locally strongly accessible from every $(\eta, v) \in \mathbb{R}^6$, and as the vector fields f, g_1, g_2 are real-analytic, this implies that the Lie algebra rank condition is satisfied $\forall (\eta, v) \in \mathbb{R}^6$ (Sussmann 1987). We note from the proof of Lemma 1 that any bracket with degree greater than 4 can be expressed as a linear combination of lower order brackets. We also note that the degree of 'bad' brackets must be odd (Sussmann 1987). The 'bad' bracket of degree 1 is f, which vanishes at any equilibrium. The 'bad' brackets of degree 3 are $[g_1, [f, g_1]]$ and $[g_2, [f, g_2]]$ which are identically zero vector fields. Consequently, the system (14)

satisfies the Sussmann Theorem, and is thus small time locally controllable from any equilibrium (Sussmann 1987).   □

The following proposition shows that a time-varying feedback law that is non-smooth at the origin, can asymptotically stabilize the surface vessel. An almost smooth periodic time-varying feedback law for the system (14) is a function τ in $C^0(\mathbb{R}^6 \times \mathbb{R}; \mathbb{R}^2)$ of class C^∞ on $(\mathbb{R}^6 \backslash \{0\}) \times \mathbb{R}$ that satisfies

$$\tau(0, t) = 0 \qquad \forall t \in \mathbb{R} \tag{15}$$

$$\tau(\eta, v, t + T) = \tau(\eta, v, t) \qquad \forall t \in \mathbb{R} \tag{16}$$

Proposition 3 *The underactuated surface vessel (14) is locally asymptotically stabilizable in small time by means of an almost smooth periodic time-varying feedback law.*

Proof. By (Coron 1995), as the dimension of the vector (η, v) is more than 4 and the vector fields f, g_1, g_2 are analytic, this follows from Proposition 2.   □

4. Exponential stabilization of the underactuated surface vessel

From Proposition 3 we know that the surface vessel can be asymptotically stabilized by an almost smooth periodic time-varying feedback law. However it is not trivial to find such a feedback law. M'Closkey and Murray (1993) and Morin and Samson (1995) have developed important analysis tools for continuous systems when the vector fields are homogeneous with respect to a given dilation. The drift vector field of the surface vessel (14) however includes trigonometric terms, and is thus not homogeneous with respect to any dilation.

However, the homogeneity property is coordinate dependent. Motivated by this we seek a coordinate transformation that renders the system homogeneous. We propose the following coordinate transformation, which is a global diffeomorphism:
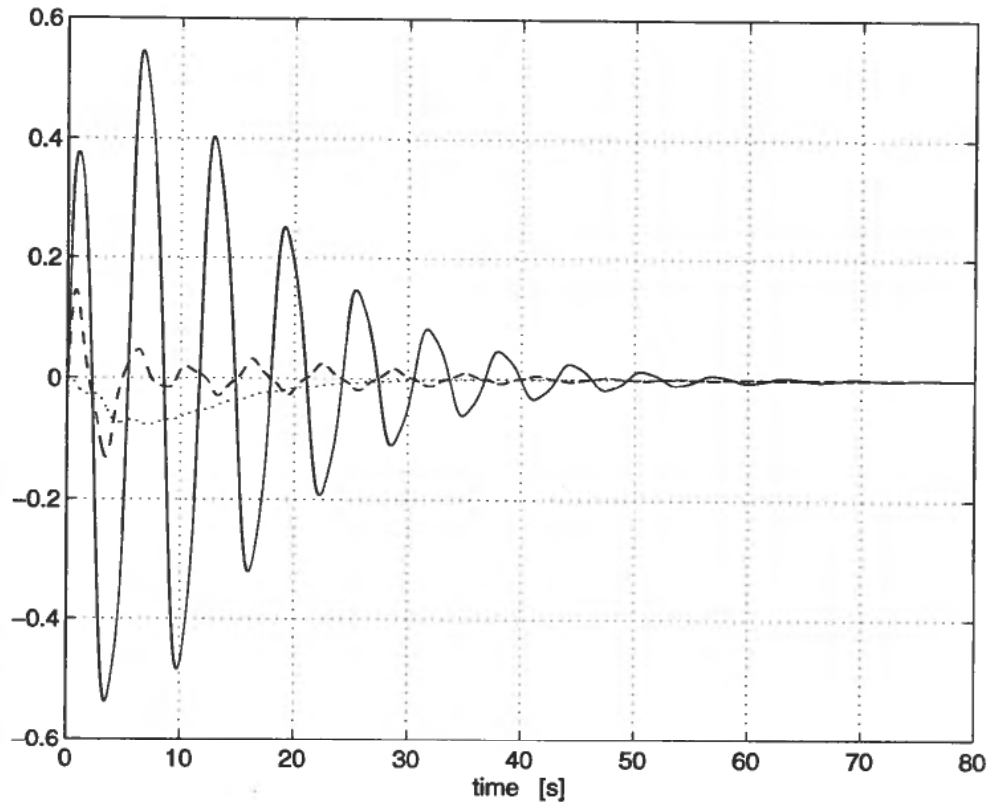
$$\begin{aligned} z_1 &= \cos(\psi)x + \sin(\psi)y \\ z_2 &= -\sin(\psi)x + \cos(\psi)y \\ z_3 &= \psi \end{aligned} \tag{17}$$

The state equations of the surface vessel are then:

$$\begin{aligned} \dot{z}_1 &= u + z_2 r \\ \dot{z}_2 &= v - z_1 r \\ \dot{z}_3 &= r \\ \dot{u} &= \frac{m_{22}}{m_{11}} vr - \frac{d_{11}}{m_{11}} u + \frac{1}{m_{11}} \tau_1 \\ \dot{v} &= -\frac{m_{11}}{m_{33}} ur \frac{m_{11}}{m_{22}} ur - \frac{d_{22}}{m_{22}} v \\ \dot{r} &= \frac{m_{11} - m_{22}}{m_{33}} uv - \frac{d_{33}}{m_{33}} r + \frac{1}{m_{33}} \tau_3 \end{aligned} \tag{18}$$

Proposition 4 *Consider the functions*
$$u_d(z, v, t) = -k_1 z_1 + \rho^\alpha(z, v)\sin(t/\varepsilon)$$

$$\tag{19}$$

$$r_d(z, v, t) = -k_2 z_3 + \frac{2}{\rho^\alpha(z, v)}(kz_2 + dv)\sin(t/\varepsilon)$$

Figure 3. Velocities $u(-)$, $v(\cdots)$, $r(--)$.

where $k_1, k_2 > 0$, $k > 0$, $d > 0$ and $\rho^\alpha(z, v)$ is any homogeneous norm associated with the dilation $\delta_\lambda^a(z, v, t) = (\lambda z_1, \lambda^2 z_2, \lambda z_3, \lambda^2 v, t)$. Given the following continuous time-varying feedback law

$$\tau_1(z, v, t) = -(k_3 m_{11} - d_{11})u + k_3 m_{11} u_d(z, v, t)$$

$$\tau_3(z, v, t) = -(k_4 m_{33} - d_{33})r + k_4 m_{33} r_d(z, v, t)$$

(20)

Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for positive and large enough parameters k_3, k_4, the feedback law (20) locally exponentially stabilizes the origin of the system (14).

Proof. The system (18)–(20) may be written

$$\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = l(z, v, t) + h(z, v, t)$$

(21)

where

$$l(z, v, t) = \begin{bmatrix} u \\ v - z_1 r \\ r \\ \frac{1}{m_{11}}(\tau_1 - d_{11}u) \\ -\frac{m_{11}}{m_{22}}ur - \frac{d_{22}}{m_{22}}v \\ \frac{1}{m_{33}}(\tau_3 - d_{33}r) \end{bmatrix}$$

(22)

$$
h(z, v, t) = \begin{bmatrix} z_2 r \\ 0 \\ 0 \\ \frac{m_{22}}{m_{11}} vr \\ 0 \\ \frac{m_{11} - m_{22}}{m_{33}} uv \end{bmatrix}
\tag{23}
$$

Consider the dilation

$$
\delta_\lambda^\beta(z, v, t) = (\lambda z_1, \lambda^2 z_2, \lambda z_3, \lambda u, \lambda^2 v, \lambda r, t)
\tag{24}
$$

As the functions τ_1 and τ_3 are homogeneous of degree 1 with respect to the dilation, and continuous for $(z, v) \neq 0$, they are also continuous at zero. The vector function l is thus continuous. It is furthermore time-periodic and $l(0, 0, t) = 0$. The vector function h is continuous. The system

$$
\begin{bmatrix} \dot z \\ \dot v \end{bmatrix} = l(z, v, t)
\tag{25}
$$

is homogeneous of degree 0 with respect to the dilation δ_λ^β, and the vector field h is homogeneous of degree 2 with respect to δ_λ^β. Thus the solution $(z, v) = (0, 0)$ of the system (21) is locally ρ^β-exponentially stable if the equilibrium $(z, v) = (0,0)$ is a locally asymptotically stable equilibrium of the system (25) (Morin and Samson, Prop. 2).

We reduce the system (25), by defining $u_d \overset{\text{def}}{=} u$ and $r_d \overset{\text{def}}{=} r$ as control variables.

$$
\begin{bmatrix} \dot z \\ \dot v \end{bmatrix} = \begin{bmatrix} u_d \\ v - z_1 r_d \\ r_d \\ -\frac{m_{11}}{m_{22}} u_d r_d - \frac{d_{22}}{m_{22}} v \end{bmatrix}
\tag{26}
$$

With controls given by (19) the averaged system of (26) is

$$
\begin{bmatrix} \dot z \\ \dot v \end{bmatrix} = \begin{bmatrix} -k_1 z_1 \\ v + k_2 z_1 z_3 \\ -k_2 z_3 \\ -\frac{m_{11}}{m_{22}} k_1 k_2 z_1 z_3 - \frac{m_{11}}{m_{22}}(k z_2 + dv) - \frac{d_{22}}{m_{22}} v \end{bmatrix}
\tag{27}
$$

The linearization of (27) about $(z, v) = (0, 0)$ is obviously locally asymptotically stable. Furthermore the system (26) is continuous, time-periodic and homogeneous of degree 0 with respect to the dilation δ_λ^α. Thus there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the origin of (26) is locally exponentially stable (M'Closkey and Murray 1993, Th. 4.1).

The functions $u_d(z, v, t)$ and $r_d(z, v, t)$ are continuous, time-periodic, differentiable with respect to t, to class C^1 on $(\mathbb{R}^3 \times \mathbb{R} - \{0, 0\}) \times \mathbb{R}$, and homogeneous of degree 1 with respect to the dilation δ_λ^α. The equations for u and r in (25) with control (20) are

$$
\dot u = -k_3(u - u_d(z, v, t))
\tag{28}
$$

$$
\dot r = -k_4(r - r_d(z, v, t))
\tag{29}
$$

Thus for positive and large enough values of k_3, k_4 the origin $(z, v) = (0, 0)$ of the system
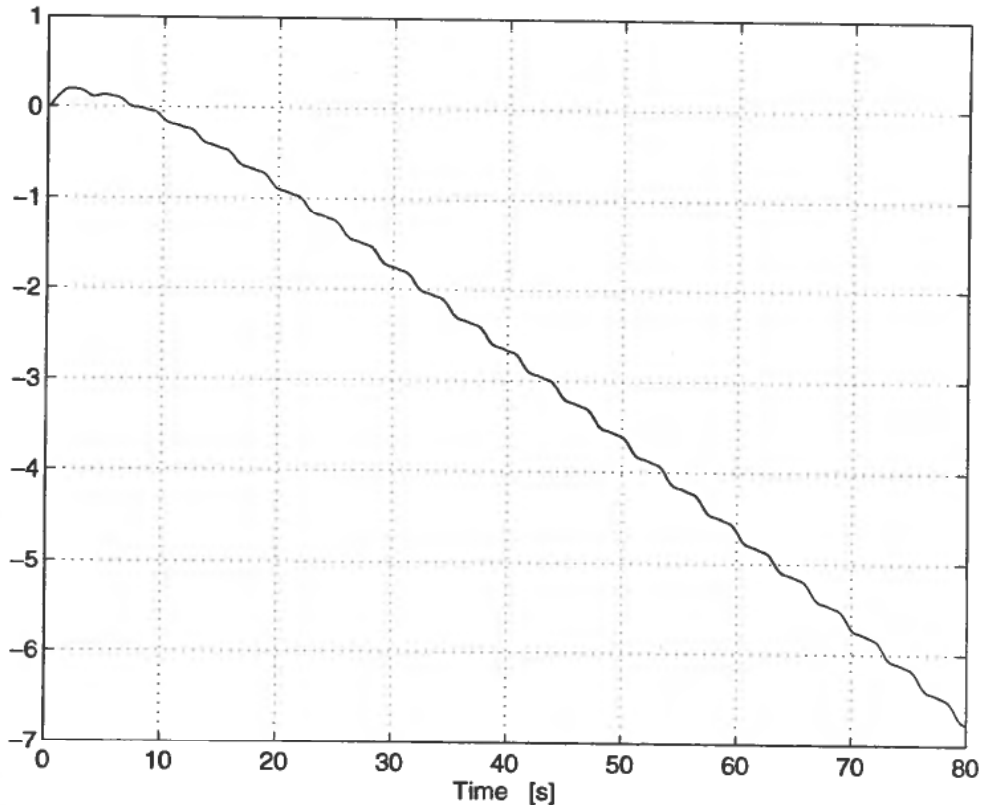
Figure   4.   The   natural   logarithm   of   the   homogeneous   norm $\rho^\beta(z, v) = \sqrt{z_1^2 + |z_2| + z_3^2 + u^2 + |v| + r^2}$.

(25) with control (19)–(20) is locally asymptotically stable (Morin and Samson 1995, Cor. 1).

Thus $(z, v) = (0, 0)$ is a locally exponentially stable equilibrium of (21) (Morin and Samson 1995, Prop. 2). As (17) is a diffeomorphism, $(\eta, v) = (0, 0)$ is therefore a locally exponentially stable equilibrium of (14)–(20).   □

Remark 1   The controller in Proposition 4 also exponentially stabilizes surface vessels with higher order damping, i.e.

$$D(v) = \text{diag}\{d_{11} + \sum_{i \geq 2} d_{u^i}|u|^{i-1}, \; d_{22} + \sum_{i \geq 2} d_{v^i}|v|^{i-1}, \; d_{33} + \sum_{i \geq 2} d_{r^i}|r|^{i-1}\}$$

This is easily shown by including the higher order terms into $h(z, v, t)$.

5.   Simulations

The action of the control law (19)–(20) has been simulated with control parameters

$$k_1 = 1.5, \; k_2 = 1.5, \; k_3 = 10, \; k_4 = 10$$
$$k = 0.2, \; d = 1.7, \; \varepsilon = 1$$

The homogeneous norm used in the control law was

$$\rho^\alpha(z, v) = \sqrt{z_1^2 + |z_2| + z_3^2 + |v|} \tag{31}$$

and the initial conditions were:

$$[x(0), y(0), \psi(0), u(0), v(0), r(0)]^T = [0, 1, 0, 0, 0, 0]^T$$

The feedback law (19)–(20) make the origin of the underactuated surface vessel (14) exponentially stable for small enough values of ε and large enough values of k_3 and k_4. The simulations illustrate that ε does not have to be very small nor k_3 or k_4 very large.

We see that the natural logarithm of the homogenous norm $\rho^\beta(z, v)$ is upper bounded by a decreasing straight line. This illustrates the exponential convergence of the system to zero.

We note from the analysis that the only model parameters that the performance of the controller may depend on exact knowledge of, are d_{11} and d_{33}. We have simulated the action of the controller when the terms $d_{11}u$ and $d_{33}r$ in (20) are neglected, and also when d_{11} and d_{33} have much larger values than the known model parameters, and the controller still provides exponential stability. This indicates that the controller is robust against model parameter uncertainty.

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