MODELING, IDENTIFICATION AND CONTROL, 1993, VOL. 14, NO. 4, 219–227
doi:10.4173/mic.1993.4.3

Design of the property transformation in elementary nonlinear decoupling of multivariable processes

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Keywords: *Nonlinear control, nonlinear decoupling, multivariable process, property transformation.*

In elementary nonlinear decoupling theory a property space (z) is defined as a transformation of the state space (x). Elementary nonlinear decoupling has the purpose of generating the control vector u which drives the system in such a way that the property vector has a desired rate of change. A method is described in this paper for the design of this property transformation which makes the nonlinear decoupling realizable and at the same time makes the system satisfy certain dynamic specifications.

1. Elementary nonlinear decoupling (END)

The process to be controlled is in general described by a nonlinear model where

$$\dot{x} = f(x, u, v) \tag{1}$$

x state vector ($\dim x = n$)

u control vector ($\dim u = r$)

v disturbance vector ($\dim v = p$)

$f(\cdot)$ vector of nonlinear functions ($\dim f = n$)

In the following we shall assume for simplicity that the model is linear in u so that (1) takes the form

$$\dot{x} = f(x, v) + B(x)u \tag{2}$$

This simplifies the arguments but the general solution based upon (1) is easily available (Balchen 1991).

The principle of *elementary nonlinear decoupling* expresses that we desire to find the control vector (u) which will drive the process (2) so that some property vector defined by $z = g(x)$ will attain a specified rate $\dot{z} = \dot{z}_d$.

The elementary solution to this problem is

$$u = \left(\frac{\partial g(\cdot)}{\partial x} B(x)\right)^{-1} \left(\dot{z}_d - \frac{\partial g(\cdot)}{\partial x} f(x, v)\right) \tag{3}$$

provided the inverse $((\partial g/\partial x)B(x))^{-1}$ exists and the system is stable.

Received 9 June 1993.

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Extended version of paper no. 126b presented at the AIChE Annual Meeting in Miami Beach, November 1992.

An alternative approach to nonlinear decoupling is developed on the basis of *differential geometry* and often referred to as *exact linearization* (Isidori *et al.* 1981, Isidori 1989). This approach, though more general, yields a result which is hard to realize in some practical systems because *Lie derivatives* of high order must be computed. In implementations it is preferable to *integrate* rather than *differentiate* noise corrupted signals. The major difference between *exact linearization* and *elementary nonlinear decoupling* is that the latter is based on the *design of a property transformation* which yields a *directly invertible system.*

A number of extensions and applications of elementary nonlinear decoupling are reported in Balchen, Lie and Solberg (1988), Lie and Balchen (1988), Lie and Balchen (1990) and Balchen (1991). A comprehensive review of literature in the general field of nonlinear control is given in Henson and Seborg (1991).

In most cases the property transformation will be chosen linear so that

$$z = Dx \tag{4}$$

where D is a constant matrix. (3) expresses the *necessary conditions* for elementary nonlinear decoupling to exist. As can be seen from (3), the whole state (x) must be available in order to compute u. A block diagram illustrating the solution of (3) is shown in Fig. 1.

The new input $\dot{z}_d$ is seen to drive m independent integrators which can be controlled by a simple diagonal control matrix (G) towards the setpoints z_0. In case the system is of the general form (1) the solution will take the form as shown in Fig. 2. In order to arrive at *sufficient* conditions for elementary nonlinear decoupling one must consider also that the final system as shown in Fig. 2

- must be stable and
- must show acceptable behaviour in the original state space (x) under normal disturbances.

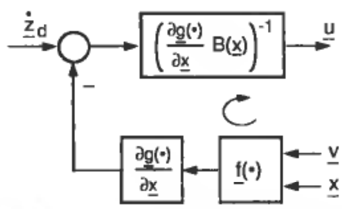

Figure 1. Block diagram of END solution based on (2).

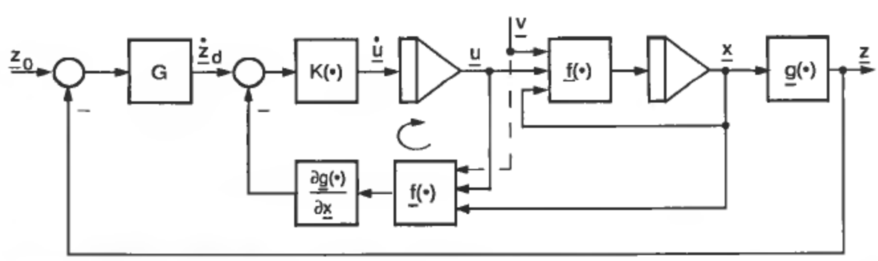

Figure 2. Block diagram of extended END solution with control system based on (1).

Even though the resulting system with nonlinear decoupling with respect to the property vector (z) with $\dim z = m \leqslant \dim \boldsymbol{u} = r$ has order m the total system is still of order $n = \dim \boldsymbol{x}$. The apparent reduction in order $m \leqslant n$ of the input–output relation is caused by what in linear systems is known as pole-zero-cancellation.

1.1. *SISO example of elementary nonlinear decoupling*

A very simple example of a linear SISO system will illustrate this point. Consider the system on *companion* form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}u \tag{5}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_1 & -a_2 & -a_3 & \dots & a_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_n \end{bmatrix}$$

$$z = [d_1 \dots d_n]\boldsymbol{x} = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} \tag{6}$$

The first condition for elementary decoupling of this system is

$$\boldsymbol{d}^{\mathrm{T}}\boldsymbol{b} \neq 0$$

which leads to $d_n \neq 0$. In other words it appears at this point that the coefficients $d_1, d_2, \dots d_{n-1}$ may be given any numerical values as long as $d_n \neq 0$. An easy way to see the consequence of this is to determine the transfer function from u to z as

$$\frac{z}{u}(s) = \frac{d_n s^{n-1} + \dots d_2 s + d_1}{s^n + a_n s^{n-1} + \dots a_2 s + a_1} b_n \tag{7}$$

As can be seen, the system is of order one. Dividing by s^{n-1} (7) can be written

$$\frac{z}{u}(s) = \frac{d_n + d_{n-1}s^{-1} + \dots + d_1 s^{-(n-1)}}{s + a_n + a_{n-1}s^{-1} + \dots + a_1 s^{-(n-1)}} b_n \to \frac{1}{s + a_n} d_n b_n \quad |s| \to \infty \tag{8}$$

which again shows that the system is of order one.

The property vector defined by (6) can be interpreted as a linear combination of the process output (x_1) and all its derivatives up to the $(n-1)$th. Controlling the quantity x_1 alone is thus not possible by elementary nonlinear decoupling.

2. Design of the property space

When the necessary conditions for elementary nonlinear decoupling are satisfied, it is still of interest to determine ways to specify the *property space* such that certain goals are reached.

Considering the nonlinear system of (2) the control vector realizing elementary nonlinear decoupling will be given by (3) which when applied to (2) yields

$$\dot{\boldsymbol{x}} = \left(I - B(\boldsymbol{x})\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} B(\boldsymbol{x})\right)^{-1} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right)\boldsymbol{f}(\boldsymbol{x}) + B(\boldsymbol{x})\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} B(\boldsymbol{x})\right)^{-1} \dot{\boldsymbol{z}}_d \tag{9}$$

A linearized version of (9) is

$$\dot{x}=(I-B(DB)^{-1}D)Ax+B(DB)^{-1}\dot{z}_d \quad (10)$$

One of the objectives in the *design* of the property space

$$z=g(x)=Dx$$

could be to minimize some measure of the response in the state space for some given exitation of $\dot{z}_d$. One such measure may be derived by defining the transfer matrix for (10) by

$$x(s)=H_x(s)\dot{z}_d(s) \quad (11)$$

where

$$H_x(s)=(sI-(I-B(DB)^{-1}D)A)^{-1}B(DB)^{-1} \quad (12)$$

A measure of the response may be

$$L=\max{}_{\omega}\bar{\sigma}_{H_x}(j\omega) \quad (13)$$

where $\bar{\sigma}_{H_x}$ is the maximal singular value of H_x. This is the H_∞ norm. An early account of this principle is given in Balchen (1958).

The optimal choice of property space then will be found as that which minimizes L in (13), that is

$$\min_D L=\min_D(\max_\omega \bar{\sigma}_{H_x}(j\omega)) \quad (14)$$

Finding quantitative results based on these principles will be a matter of computer aided numerical computations for all but the very simplest cases. In doing this the unknown elements of the function $g(\cdot)$ (or D) should be collected in a vector α and a numerical gradient procedure applied to make $\partial L/\partial\alpha=0$.

Another way to view the problem is to specify a desirable, but possibly unrealizable property transformation

$$z^0=D^0x \quad (15)$$

and thereafter find the realizable property transformation

$$z=Dx \quad (16)$$

which minimizes some measure of the difference, $\Delta z=z-z_0$ for example

$$L=\max_\omega \bar{\sigma}_{\Delta H_z}(j\omega) \quad (17)$$

where

$$\Delta H_z(j\omega)=(D-D^0)(j\omega I-(I-B(DB)^{-1}D)A)^{-1}B(DB)^{-1} \quad (18)$$

The 'optimal' property space would then be found by

$$\min_D L=\min_D(\max_\omega \bar{\sigma}_{\Delta H_z}(j\omega)) \quad (19)$$

By this optimization procedure presumably the best approximation to the desired property transformation is obtained. Now a feedback system controlling z can be realized as shown as the inner loop in Fig. 3 with z_0 as reference input. This loop is decoupled and linearized. A new feedback system with z^0 as the output can now be established as shown in Fig. 3. This loop is only approximately decoupled and linear, but due to the optimization procedure described above, the deviation should be small

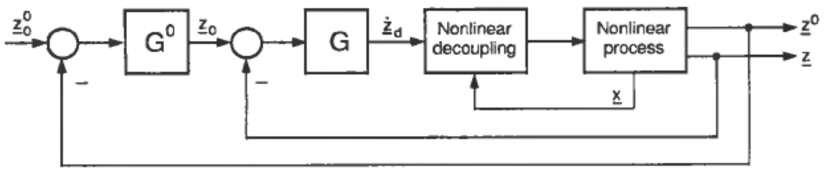

Figure 3. Block diagram of approximately decoupled control system.

and thus the desired performance achieved. A detailed analysis of the errors involved is both feasible and desirable.

But the transfer matrix $\Delta H_z(s)$ will have specific dynamic properties that are only functions of the D and D^0 matrices and not the A matrix because this matrix disappears in the algebraic manipulations. This does not make sense in physical reality. Hence in order for the resulting system to make physical sense we must specify something about the dynamics of $\Delta H_z(s)$ relative to the dynamics of the original system, say, $\phi(s)=(sI-A)^{-1}$ or the eigenvalues of A. A reasonable specification of the dynamics of $\Delta H_z(s)$ is to relate a characterizing parameter to the largest eigenvalue of A. If the parameter is a frequency it could be

$$\omega_{\Delta H_z}=k|\lambda_{A\,\max}|$$

The factor k gives a measure of the bandwidth of the decoupled system relative to the bandwidth of the original process.

2.1. *SISO example of design of dynamics*

In the example above with a SISO process of third order the desirable property transformation is suggested to be

$$z^0=\boldsymbol{d}^{0\mathrm{T}}\boldsymbol{x}=[1\,0\,0]\boldsymbol{x}$$

and the realizable transformation

$$z=[1\;d_2\;d_3]\boldsymbol{x}$$

and

$$\Delta H_z(s)=\Delta h_z(s)=(\boldsymbol{d}^{\mathrm{T}}-\boldsymbol{d}^{0\mathrm{T}})(sI-(I-\boldsymbol{b}(\boldsymbol{d}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{d})A)^{-1}\boldsymbol{b}(\boldsymbol{d}^{\mathrm{T}}\boldsymbol{b})^{-1} \tag{20}$$

$$=\frac{d_2+d_3s}{1+d_2s+d_3s^2}=d_2\cdot\frac{1+\dfrac{s}{\omega_1}}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}\rightarrow\frac{1}{s}\quad\text{for } |s|\rightarrow\infty$$

Since this case is scalar

$$\bar{\sigma}_{\Delta h_z}(j\omega)=|\Delta h_z(j\omega)|$$

In Fig. 4 $|\Delta h_z(j\omega)|$ is shown for different d_2 and d_3. The $|\Delta h_z(j\omega)|_{\max}$ will occur near the resonance. If, for simplicity the relative damping ζ is assumed to be $\zeta=0{\cdot}5$ we get $\omega_1=\omega_0$ and $d_3=d_2^2$ and therefore

$$\Delta h_z(s)=d_2\cdot\frac{1+d_2s}{1+d_2s+(d_2s)^2} \tag{21}$$

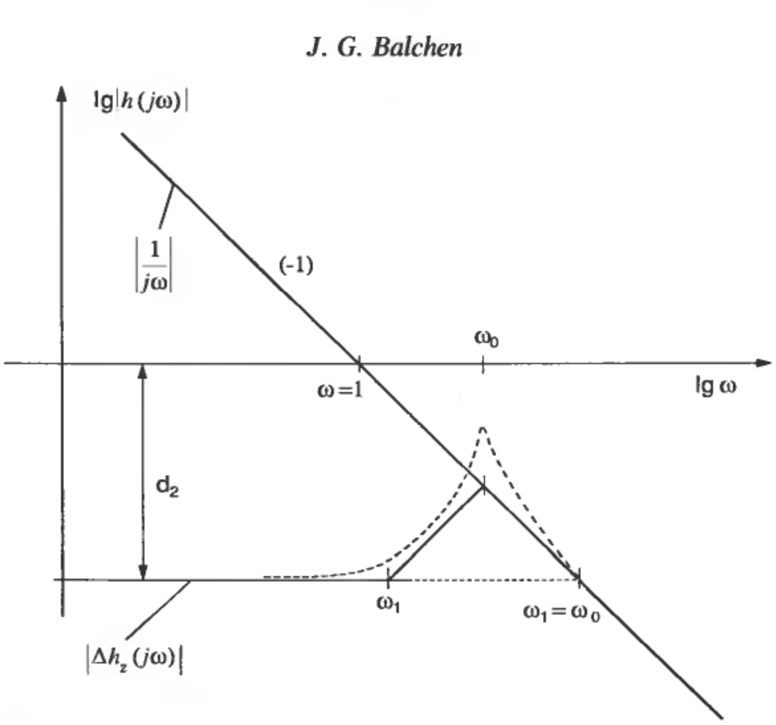

Figure 4. Frequency response of error function in approximate nonlinear decoupling.

According to the argument above we shall specify

$$\frac{1}{d_2}=k|\lambda_{A\,\max}|$$

where we could choose, say, $k=10$. If $\lambda_{A\,\max}=10$ we thus get $d_2=10^{-2}$ and $d_3=10^{-4}$. This set of parameters defines the property-space-transformation

$$z=[d_1 d_2 d_3]\boldsymbol{x}=[1\;10^{-2}\;10^{-4}]\boldsymbol{x}$$

The transfer function for the realizable system is

$$\frac{z}{\dot{z}_d}=\frac{1}{s} \tag{22}$$

and the transfer function for the decoupled 'desirable' system

$$\frac{x_1}{\dot{z}_d}(s)=\frac{1}{s(1+d_2 s+(d_2 s)^2)}=\frac{1}{s\left(1+2\cdot 0{\cdot}5\cdot\frac{s}{100}+\left(\frac{s}{100}\right)^2\right)} \tag{23}$$

The difference between these two transfer functions is, as decided, only the double pole at frequency $\omega_0=100$ which is 10 times higher than the highest eigenvalue of A. Had we chosen $k=100$ the resulting property transformation would have been

$$z=[1\;10^{-3}\;10^{-6}]\boldsymbol{x}$$

In the eqn (3) determining $\boldsymbol{u}$ it is clearly seen what the consequence is of reducing d_3. The 'gain' of the equation is $(\boldsymbol{d}^{\mathrm{T}}\boldsymbol{b})^{-1}=1/d_3 b$ which for the last case will give $1/d_3 b=10^6$ (assuming $b=1$). With such a high 'gain' in the inversion equation saturation will occur. Hence the factor k must be given a smaller numerical value based on process related considerations.

2.2. *Example of more complex nonlinear process*

To further illustrate the techniques of elementary nonlinear decoupling, a more complex nonlinear process presented in Kravaris and Soroush (1988) is considered.

The process is described by the nonlinear differential equations

$$\begin{aligned}\dot{x}_1 &= -x_3 + x_1 u_1 \\ \dot{x}_2 &= x_1^2 - 2x_1 u_1 \\ \dot{x}_3 &= -x_3 x_1 + x_4 + (1+x_1)x_1 u_1 \\ \dot{x}_4 &= u_2\end{aligned} \tag{24}$$

The desirable, but nonrealisable property to be controlled is

$$z^0 = D^0 x = \begin{bmatrix} 1000 \\ 0100 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{25}$$

Inspection of (24) reveals immediately that the state x_4 can be removed because it can be controlled directly with a feedback of any bandwidth through the control variable u_2. Thus we can replace

$$x_4 = u_2 \tag{26}$$

and get a reduced system of three differential equations

$$\begin{aligned}\dot{x}_1 &= -x_3 + x_1 u_1 \\ \dot{x}_2 &= x_1^2 - 2x_1 u_1 \\ \dot{x}_3 &= -x_3 x_1 + u_2 + (1+x_1)x_1 u_1\end{aligned} \tag{27}$$

Furthermore by inspection we observe that the control variable u_1 always appears in a product with the state variable x_1. This obviously must lead to a special situation when $x_1 = 0$.

Introducing the property transformation

$$z = Dx = \begin{bmatrix} 1 & d_{12} & d_{13} \\ d_{21} & 1 & d_{23} \end{bmatrix} x \tag{28}$$

based on the reduced state of (27), and the compact form (2)

$$\dot{x} = f(x) + B(x)u \tag{29}$$

where

$$f(x) = \begin{bmatrix} -x_3 \\ x_1^2 \\ -x_3 x_1 \end{bmatrix} \tag{30}$$

and

$$B(x) = \begin{bmatrix} x_1 & 0 \\ -2x_1 & 0 \\ (1+x_1)x_1 & 1 \end{bmatrix} \tag{31}$$

we want to compute the expression

$$(DB(x))^{-1}=\left(\begin{bmatrix}1 & d_{12} & d_{13}\\ d_{21} & 1 & d_{23}\end{bmatrix}\begin{bmatrix}x_1 & , & 0\\ -2x_1 & , & 0\\ (1+x_1)x_1 & , & 1\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}x_1(1-2d_{12}+d_{13}(1+x_1)) & , & d_{13}\\ x_1(d_{21}-2+d_{23}(1+x_1)) & , & d_{23}\end{bmatrix}^{-1} \tag{32}$$

$$\frac{1}{x_1(d_{23}-2d_{23}d_{12}-d_{13}d_{21}-2d_{13})}\begin{bmatrix}d_{23} & , & -d_{13}\\ -x_1(d_{21}-2+d_{23}(1+x_1)) & , & x_1(1-2d_{12}+d_{13}(1+x_1))\end{bmatrix}$$

We also want to compute

$$Df(x)=\begin{bmatrix}1 & d_{12} & d_{13}\\ d_{21} & 1 & d_{23}\end{bmatrix}\begin{bmatrix}-x_3\\ x_1^2\\ -x_3x_1\end{bmatrix}$$

$$=\begin{bmatrix}-x_3+d_{12}x_1^2-d_{13}x_3x_1\\ -d_{21}x_3+x_1^2-d_{23}x_3x_1\end{bmatrix} \tag{33}$$

The algorithm for computing the control vector realizing elementary nonlinear decoupling, is then given by applying (32) and (33) to

$$u=(DB(x))^{-1}(\dot{z}_d-Df(x)) \tag{34}$$

As can be seen from the numerator of the first term in the final expression of (32) we may choose all the elements d_{12}, d_{21}, d_{13} or d_{12}, d_{21}, d_{23} equal to zero without (DB) becoming singular. Thus as a very simple illustrative example let $d_{12}=d_{21}=d_{13}=0$ and $d_{23}\neq 0$. Then we get from (34)

$$u_1=\frac{1}{x_1}(\dot{z}_{d1}+x_3)$$

$$u_2=\left(\frac{2}{d_{23}}-(1+x)\right)\dot{z}_{d1}+\frac{1}{d_{23}}\dot{z}_{d2}+\left(\frac{2}{d_{23}}-1\right)x_3-\frac{1}{d_{23}}x_1^2 \tag{35}$$

which when applied to (27) yields

$$\dot{x}_1=\dot{z}_{d1}$$

$$\dot{x}_2=\dot{z}_{d2}-d_{23}\dot{x}_3 \tag{36}$$

$$\dot{x}_3=\frac{1}{d_{23}}[2\dot{z}_{d1}+\dot{z}_{d2}+2x_3-x_1^2]$$

This illustrates that we have achieved approximate decoupling because

$$\dot{x}_1=\dot{z}_{d1}$$

$$\dot{x}_2\approx\dot{z}_{d2} \tag{37}$$

3. Applications

Elementary nonlinear decoupling is a model based technique particularly effective for the design of control laws for highly nonlinear processes. Such processes are known to occur frequently in reactor technology and separation technology in the chemical industry, as well as in kinematic-dynamic systems of robotics and vehicles. An introductory study of the application of elementary nonlinear decoupling to composition control of a distillation column has given promising results (Skarstein 1992).

4. Conclusion

A systematic procedure is presented for the design of the property space for elementary nonlinear decoupling. The procedure leads to a decoupling strategy which is both theoretically and physically appealing.

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