MODELING, IDENTIFICATION AND CONTROL, 1991, VOL. 12, NO. 3, 149–158
doi:10.4173/mic.1991.3.5

Maximal imaginary eigenvalues in optimal systems‡

DAVID DI RUSCIO†

Keywords: *Linear optimal control, pole placement, eigenvalues, multivariable control systems, control system design.*

In this note we present equations that uniquely determine the maximum possible imaginary value of the closed loop eigenvalues in an LQ-optimal system, irrespective of how the state weight matrix is chosen, provided a real symmetric solution of the algebraic Riccati equation exists. In addition, the corresponding state weight matrix and the solution to the algebraic Riccati equation are derived for a class of linear systems. A fundamental lemma for the existence of a real symmetric solution to the algebraic Riccati equation is derived for this class of linear systems.

1. Introduction

Application of the LQ approach to regulator design involves choosing the state and control input weighting matrices, $\mathbf{Q}$ and $\mathbf{P}$, that provide satisfactory closed-loop performance. The closed loop performance is related to the locations of the closed loop eigenvalues. In some applications we are not interested in the exact locations of the closed loop eigenvalues, rather the region where the eigenvalues are located. For example, approximate knowledge of the locations of the closed loop eigenvalues is required when designing the performance weights. The problem of determining the region where the closed loop eigenvalues are located is the topic of this paper.

We know that for a suitable choice of performance weighting matrices the real part of the closed loop eigenvalues can be located all along the negative real axis in the complex plane. This means that the real part of the closed loop eigenvalues is not bounded. This is generally not the case for the imaginary parts of the closed loop eigenvalues.

In this paper we will show and determine an exact upper bound for the imaginary part of the closed loop eigenvalues for a class of linear systems. In addition, the corresponding 'worst case' state weight matrix and the solution to the algebraic Riccati equation are derived. A fundamental lemma for the existence of a real symmetric solution to the algebraic Riccati equation is derived for this class of linear systems.

The rest of the paper is organized as follows. Section 2 presents the problem definitions, i.e. determines the maximum possible imaginary eigenvalue in an LQ-optimal system, the corresponding state weight matrix $\mathbf{Q}$, and the corresponding solution to the algebraic Riccati equation. Once the maximum imaginary eigenvalue is determined, we will show that the closed loop eigenvalues are located inside a horizontal strip in the complex plane, no matter how the state weight matrix is chosen, provided a real symmetric solution of the algebraic Riccati equation exists. The problem solution is stated in Section 3. Numerous examples are given in Section 4 and some concluding remarks follow in Section 5.

Received 27 February 1991.

† Division of Engineering Cybernetics, Norwegian Institute of Technology, N-7034 Trondheim, Norway.

‡ This paper was presented at the European Control Conference ECC '91, Grenoble, France, 2–5 July 1991, and is reprinted with permission.

2. Problem formulation

Consider the linear, time-invariant, dynamic system

$$\dot{x} = Ax + Bu \tag{1}$$

where **(A, B)** is a stabilizable pair, and the quadratic objective functional of long or infinite settling time

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T P u)\, dt \tag{2}$$

where x is an n-dimensional state vector, u is an r-dimensional control input vector, **A** and **B** are constant matrices of appropriate dimensions and **Q** and **P** are $n \times n$ symmetrical and $r \times r$ positive definite matrices respectively. It is assumed that $(Q^{1/2}, A)$ is a detectable pair. The optimal control that minimizes criterion (2) and the corresponding closed loop system is given by

$$u = Gx, \qquad G = -P^{-1}B^T R \tag{3}$$

$$\dot{x} = (A + BG)x = (A - HR)x \tag{4}$$

where **H** is given by (6) and **R** is a solution to the algebraic Riccati equation (ARE)

$$-\dot{R} = A^T R + RA - RHR + Q = 0 \tag{5}$$

$$H = BP^{-1}B^T \tag{6}$$

The ARE (5) can have solutions which are real or complex, symmetric or non-symmetric, definite or indefinite and the set of solutions can be either finite or infinite. See, among others Kučera (1989), Lancaster and Rodman (1980) and Willems (1971). This paper is restricted to real symmetric solutions of the ARE.

Both the closed loop system eigenvalues, and the solution **R** to the ARE can be determined from the state/co-state system matrix (7), Laub (1979), Di Ruscio and Balchen (1990). The state/co-state system matrix (the Hamiltonian matrix) is derived from optimal control theory by augmenting the co-state equation $\dot{p} = -Qx - A^T p$ to the state space model (1), with the optimal control input vector $u = -P^{-1}B^T p$. The Hamiltonian is

$$F = \begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix} \in \mathbf{R}^{2n \times 2n} \tag{7}$$

We will show that when $H > 0$ then the closed loop eigenvalues are bounded by the region shown in Fig. (1), irrespective of how the state weight matrix Q is chosen,

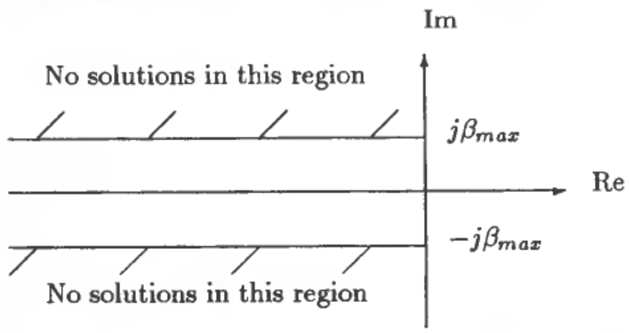

Figure 1. Region where the closed loop eigenvalues are located.

provided there is a real symmetric solution of the ARE. Thus the problem is to determine the maximum possible imaginary value β_{max}, the corresponding 'worst case' state weight matrix **Q**, and the corresponding real symmetric solution **R** to the ARE. β_{max}, **Q** and **R** are extreme solutions to the LQ-problem. Note that $H>0$ only when $\dim(\boldsymbol{x})=\dim(\boldsymbol{u})=n$.

3. Main results

The main results in this section are stated in three lemmas. Lemma 1 consists of a necessary and sufficient condition for the existence of a real symmetric solution to the ARE. Lemmas 2 and 3 state analytical solutions to the n-dimensional algebraic Riccati equation where the corresponding closed-loop eigenvalues determine a finite upper bound of the imaginary value. We will prove that the imaginary part of these closed loop eigenvalues is the maximum possible in an LQ-optimal system, no matter how the state weight matrix **Q** is chosen, provided there is a real symmetric solution of the ARE.

Lemma 1

When $H=H^{T}>0$, then there is a real symmetric solution **R** to the algebraic Riccati Equation (5) having the property $\mathrm{Re}\lambda(A-HR)\leqslant 0$, if and only if, the symmetrical state weight matrix **Q** satisfies

$$Q-Q_0-X(\omega)\geqslant 0, \quad \forall\omega\in R^1 \tag{8}$$

where

$$\begin{aligned} Q_0 &= -(H^{-1}Z_0^2+A^{T}H^{-1}A) \\ &= Z_0^{T}H^{-1}Z_0-A^{T}H^{-1}A \end{aligned} \tag{9}$$

$$\begin{aligned} X &= H^{-1}(Ij\omega-Z_0)^2 \\ &= -H^{-1}(I\omega+jZ_0)^2 \end{aligned} \tag{10}$$

$$Z_0=\tfrac{1}{2}(A-HA^{T}H^{-1}) \tag{11}$$

A proof of Lemma 1 follows immediately after the statement of Lemma 2. Note that there may be symmetric matrices **Q**, satisfying Lemma 1, which are negative semidefinite, indefinite and (of course) positive semi-definite.

Lemma 2

A unique solution to the algebraic Riccati equation with closed-loop eigenvalues at the imaginary axis is given by

$$R_0=\tfrac{1}{2}(H^{-1}A+A^{T}H^{-1}) \tag{12}$$

$$Q_0=Z_0^{T}H^{-1}Z_0-A^{T}H^{-1}A \tag{13}$$

where the closed-loop system matrix Z_0 is given by

$$Z_0=A-HR_0=\tfrac{1}{2}(A-HA^{T}H^{-1}) \tag{14}$$

The closed-loop eigenspectrum is given by

$$\lambda_i(A-HR_0)=\lambda_i(Z_0), \quad i=1,\ldots,n \tag{15}$$

The spectrum (15) is purely imaginary. This means that the eigenvalues of (14) consist of p complex-conjugate imaginary eigenvalues and s eigenvalues *in origo*, where $n=p+s$.

Proof of Lemma 1 and 2

There is a real symmetric solution to the algebraic Riccati equation having the property Re $\lambda(A-HR)\leqslant 0$, if and only if, (Lemma 5, Willems 1971),

$$[I-B^{\mathrm{T}}(-Is-A^{\mathrm{T}})^{-1}G^{\mathrm{T}}]P[I-G(Is-A)^{-1}B]=P+B^{\mathrm{T}}(-Is-A^{\mathrm{T}})^{-1}Q(Is-A)^{-1}B\geqslant 0 \tag{16}$$

for all $s=j\omega$, where $\omega\in\mathrm{R}^1$ and $G=-P^{-1}B^{\mathrm{T}}R$. Pre-multiplcation with BP^{-1} and post-multiplication with $P^{-1}B^{\mathrm{T}}$ we see that (16) is equivalent to

$$[I+H(-Is-A^{\mathrm{T}})^{-1}R]H[I+R(Is-A)^{-1}H]=H+H(-Is-A^{\mathrm{T}})^{-1}Q(Is-A)^{-1}H\geqslant 0 \tag{17}$$

Pre-multiplication with $(-Is-A^{\mathrm{T}})H^{-1}$ and post-multiplication with $H^{-1}(Is-A)$ we see that (17) is equivalent to

$$[-Is-(A-HR)^{\mathrm{T}}]H^{-1}[Is-(A-HR)]=(-Is-A^{\mathrm{T}})H^{-1}(Is-A)+Q\geqslant 0 \tag{18}$$

Equation (18) shows that for all $s=j\omega$, Q must satisfy

$$Q\geqslant -(-Is-A^{\mathrm{T}})H^{-1}(Is-A)=-(-H^{-1}s^2+2H^{-1}Z_0s+A^{\mathrm{T}}H^{-1}A) \tag{19}$$

where Z_0 is given by (14). Equation (19) can be written as

$$Q-Q_0\geqslant -(Is-Z_0)^*H^{-1}(Is-Z_0)=H^{-1}(Is-Z_0)^2=-H^{-1}(I\omega+jZ_0)^2\overset{\mathrm{def}}{=}X \tag{20}$$

where the super-index * denotes the complex conjugate, and Q_0 is given by (13). Equation (20) must be satisfied for all $s=j\omega$, where $\omega\in\mathbf{R}^1$. Note that $X=X^{\mathrm{H}}\leqslant 0$, $\forall\omega\in\mathbf{R}^1$. To show that X is Hermitian, use the fact that $Z_0=-HZ_0^{\mathrm{T}}H^{-1}$. To show that $X\leqslant 0$ we use the fact that the spectrum $\lambda(Z_0)$ is purely imaginary, $\lambda(I\omega+jZ_0)$ is real and $\lambda(I\omega+jZ_0)^2\geqslant 0$ and real. A proof of the statement that $\lambda(Z_0)$ is purely imaginary, is given in Appendix A. Note that the maximum of X is zero, in a matrix sense, and appears for all $s=\lambda_i(Z_0)$, $i=1,\ldots,n$. To show this, take the determinant of X.

From Eqn. (20) and the fact that $X\leqslant 0$ must be satisfied for all real ω we find that the state weight matrix **Q** must satisfy

$$Q-Q_0\geqslant X, \quad 0\geqslant X \tag{21}$$

and Lemma 1 is proved. From (21) we know that the extreme solution is given by $Q=Q_0$, and Eqn. (13) is proved.

An alternative way to establish this result is as follows. We are interested in extreme solutions at the imaginary axis. We then choose the equality condition of (19), i.e.

$$Q(s)=-(-H^{-1}s^2+2H^{-1}Z_0s+A^{\mathrm{T}}H^{-1}A)=Q_0+X(s) \tag{22}$$

where X is defined in (20). (22) has a minimum value, in a matrix sense, because $\partial^2Q(s)/\partial s^2=2H^{-1}\geqslant 0$. This minimum value is the solution to the problem. The minimum value, in a matrix sense, is given by

$$\frac{\partial Q(s)}{\partial s}=2H^{-1}(Is-Z_0)=0\Rightarrow s=\lambda_i(Z_0), \quad i=1,\ldots,n \tag{23}$$

Substituting the condition for a minimum, Eqn. (23), in (22) or (20) we derive Eqn. (13).

Observe that Q_0, Eqn. (9) or (13), is symmetric. The symmetry of (9) is shown by expanding the term $H^{-1}Z_0^2$ in Q_0. We have

$$Q_0 = -\tfrac{1}{4}(H^{-1}A^2 + (A^2)^T H^{-1} - H^{-1}AHA^T H^{-1} - A^T H^{-1}A) - A^T H^{-1}A \qquad (24)$$

To show that R_0 (Eqn. (12)) is the corresponding solution to the ARE, we simply write the closed loop system as

$$A - HR_0 = Z_0 = \tfrac{1}{2}(A - HA^T H^{-1}) \qquad (25)$$

Equation (25) is satisfied, if

$$R_0 = \tfrac{1}{2}(H^{-1}A + A^T H^{-1}) \qquad (26)$$

With standard algebraic manipulations it is easy to see that (13) and (12) satisfy the algebraic Riccati equation, and Lemma 2 is proved.

Lemma 3

A unique solution to the algebraic Riccati equation with real parts of the closed-loop eigenvalues equal to $-\alpha$ and constant imaginary parts equal to $\beta_i = \lambda_i(Z_0)$, $i = 1, \ldots, n$ are given by

$$R_\alpha = R_0 + \Delta R, \quad \Delta R = \alpha H^{-1} \qquad (27)$$

$$Q_\alpha = Q_0 + \Delta Q, \quad \Delta Q = \alpha^2 H^{-1} \qquad (28)$$

where R_0 and Q_0 are given in (12) and (13). The closed-loop system Z_α is given by

$$Z_\alpha = A - HR_\alpha = -\alpha I + \tfrac{1}{2}(A - HA^T H^{-1}) \qquad (29)$$

The closed-loop eigenspectrum is provided by

$$\lambda_i(A - HR_\alpha) = -\alpha + \lambda_i(Z_0), \quad i = 1, \ldots, n \qquad (30)$$

where Z_0 is defined in (14). The spectrum $\lambda(Z_0)$ is purely imaginary.

Proof of Lemma 3

The equality in (17) can be written as

$$[I + H(Is^* - A^T)^{-1}R]H[I + R(Is - A)^{-1}H] = H + H(Is^* - A^T)^{-1}Q(Is - A)^{-1}H$$
$$- 2\alpha H(Is^* - A^T)^{-1}R(Is - A)^{-1}H \geqslant 0 \qquad (31)$$

where $s = -\alpha + \sigma$ and $\sigma = j\omega$. From (31) we get

$$Q(s) \geqslant -(Is^* - A^T)H^{-1}(Is - A) + 2\alpha R \qquad (32)$$

Substituting $s = -\alpha + \sigma$ in (32) we get

$$Q(s) \geqslant -(-H^{-1}\sigma^2 + 2H^{-1}Z_0\sigma + A^T H^{-1}A) - H^{-1}\alpha^2 - (H^{-1}A + A^T H^{-1})\alpha + 2\alpha R \qquad (33)$$

We are interested in extreme solutions. We then choose the equality condition of (33)

Firstly, we optimize (33) with respect to α and get

$$\frac{\partial Q(s)}{\partial \alpha} = -2H^{-1}\alpha - (H^{-1}A + A^T H^{-1}) + 2R = 0 \qquad (34)$$

(34) is satisfied, if and only if,

$$R = \tfrac{1}{2}(H^{-1}A + A^{\mathrm{T}}H^{-1}) + \alpha H^{-1} = R_0 + \Delta R \tag{35}$$

and Eqn. (27) is deduced. Substituting (35) into (33) we get

$$Q \geqslant Q_\alpha + H^{-1}(I\sigma - Z_0)^2 = Q_\alpha - H^{-1}(I\omega + jZ_0)^2 \tag{36}$$

Secondly, we have to optimize (36) with respect to ω. This problem is identical to proving Lemma 2, and the reader is referred to the proof of Lemma 2. A proof of the statement that the spectrum $\lambda(Z_0)$ is purely imaginary, is given in Appendix A.

We see that the extreme imaginary eigenvalue is independent of the real part α and that the results in Lemma 3, with $\alpha = 0$, are identical to the results in Lemma 2. From (34) we see that it is possible to locate the closed loop eigenvalues all along the real axis, and that the imaginary parts are limited by (29). With standard algebraic manipulations it is easy to see that (28) and (27) satisfy the ARE, Equation (5), and that (29) is the corresponding closed loop system matrix.

Note that Eqns. (17) and (31) are special cases of a more general formulation, namely

$$[I + H(S^* - A^{\mathrm{T}})^{-1}R]H[I + R(S - A)^{-1}H] = H + H(S^* - A^{\mathrm{T}})^{-1} \times (Q + S^*R + RS)(S - A)^{-1}H \tag{37}$$

where $S \in \mathbf{C}^{n \times n}$ is a complex matrix. Equation (17) is derived from (37) with $S = Is = Ij\omega$ and Eqn. (33) is derived from (37) with $S = I(-\alpha + \sigma) = I(-\alpha + j\omega)$. The closed loop system matrix $S = Z = A - HR$ satisfies Eqn. (37).

Remark 1

An LQ-optimal system, when H is non-singular, has only real eigenvalues, for all Q satisfying Lemma 2, if

$$Z_0 = 0 \Rightarrow AH - HA^{\mathrm{T}} = 0 \tag{38}$$

This means that the control input weight matrix P ($H = BP^{-1}B^{\mathrm{T}}$) can be chosen to position the imaginary parts of the closed loop eigenvalues.

Remark 2

As a consequence of the above discussion we have the maximal possible imaginary value β in an LQ-optimal system, when H is non-singular, is given by

$$\beta_{\max} = \max|\lambda_i(Z_0)|, \quad i = 1, \ldots, n \tag{39}$$

Note that the eigenvalues of Z_0 are always located at the imaginary axis. A proof is given in Appendix A.

Remark 3

Note that the maximum imaginary value of the closed loop eigenvalues is independent of a multiplicative scalar perturbation of the nominal control input weight matrix **P**, $P_\rho = \rho P$, $\rho > 0$ and $\rho \in \mathbf{R}^1$. This is seen from Eqns. (14) or (29).

Remark 4

Note that $\Delta R=\alpha H^{-1}$, $\Delta Q=\alpha^2 H^{-1}$ and $A-H\Delta R=-\alpha I+A$ in Lemma 3 also is a solution to the LQ-problem. This means that the feedback matrix $G=-\alpha P^{-1}B^{\mathrm{T}}H^{-1}$ moves the eigenvalues of A a certain amount, $-\alpha$. This result can be used, for example to determine an initial stabilizing control feedback matrix.

Remark 5

From Eqn. (18), we have that an alternative form of the ARE, when H is non-singular, is given by

$$(A-HR)^{\mathrm{T}}H^{-1}(A-HR)=A^{\mathrm{T}}H^{-1}A+Q \tag{40}$$

From (40) we have the relation

$$[\det(A-HR)]^2=\left[\prod_{i=1}^{n}\lambda_i(A-HR)\right]^2=\frac{\det(A^{\mathrm{T}}H^{-1}A+Q)}{\det(H^{-1})} \tag{41}$$

which relates the closed-loop poles to A, H and Q.

4. Numerical examples

4.1. Example 1

Consider the system (equal to Example 1 in Di Ruscio and Balchen, 1989)

$$A=\begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix} B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} P=\begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix} \tag{42}$$

$$H=BP^{-1}B^{\mathrm{T}}=\begin{bmatrix}1 & 0\\ 0 & 0{\cdot}2\end{bmatrix} \tag{43}$$

The system eigenvalues are

$$\lambda_1(A)=-2, \quad \lambda_2(A)=-1 \tag{44}$$

The extreme solutions of the LQ-problem at the imaginary axis are given by (Eqns. (13), (12) and (14))

$$Q_0=\begin{bmatrix}-7{\cdot}75 & 5{\cdot}0\\ 5{\cdot}0 & 1{\cdot}25\end{bmatrix} R_0=\begin{bmatrix}-2{\cdot}0 & 2{\cdot}5\\ 2{\cdot}5 & -5{\cdot}0\end{bmatrix} \tag{45}$$

$$Z_0=\begin{bmatrix}0{\cdot}0 & -2{\cdot}5\\ 0{\cdot}5 & 0{\cdot}0\end{bmatrix} \tag{46}$$

The eigenvalues of the optimal closed loop system Z_0 are given by

$$\lambda(Z_0)=\lambda(A-HR_0)=\pm j\beta_{\max}=\pm j\frac{5^{1/2}}{2} \tag{47}$$

The stable extreme solutions of the LQ-problem, with real parts of the closed loop eigenvalues equal to $-\alpha$, $\alpha\geqslant 0$ are given by (Eqns. (28), (27) and (29))

$$Q_\alpha=\begin{bmatrix}\alpha^2-7{\cdot}75 & 5\\ 5 & 5\alpha^2+1{\cdot}25\end{bmatrix} \tag{48}$$

$$R_\alpha=\begin{bmatrix}\alpha-2 & 2{\cdot}5\\ 2{\cdot}5 & 5\alpha-5\end{bmatrix} Z_\alpha=\begin{bmatrix}-\alpha & -2{\cdot}5\\ 0{\cdot}5 & -\alpha\end{bmatrix} \tag{49}$$

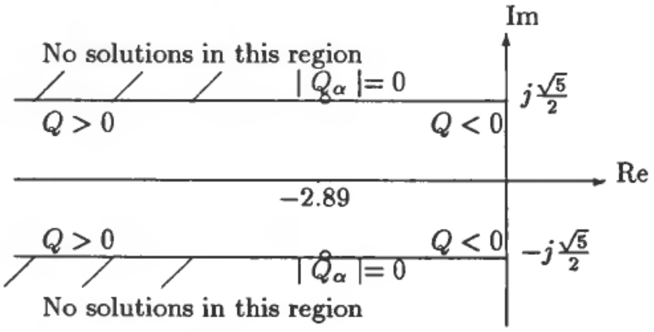

Figure 2. Region where solutions to the LQ-problem occur.

The eigenvalues of the optimal closed loop system Z_α are given by

$$\lambda(Z_\alpha)=\lambda(A-HR_\alpha)=-\alpha\pm j\beta_{\max}=-\alpha\pm j\frac{(5)^{1/2}}{2}\approx-\alpha\pm j\,1{\cdot}118 \tag{50}$$

Note that $|Q_\alpha|=0$ for $\alpha=-(\frac{15}{4}+(21)^{1/2})^{1/2}\approx-2{\cdot}89$, $Q_\alpha>0$ for $\alpha<-2{\cdot}89$ and $Q_\alpha<0$ for $-2{\cdot}89<\alpha\leqslant 0$. The region where solutions to the LQ-problem occur is illustrated in Fig. (2)

4.2. *Example 2*

Consider the same system as in Example 1, but with a free control input weight matrix **P**. The question we raise in this example is, What choice of **P** would lead to an LQ-system with only real closed loop eigenvalues, irrespective of how the state weight matrix **Q** is chosen? We get the answer to this problem from Eqn. (3).

$$Z_0=0\Rightarrow AH-HA^{\mathrm{T}}=0 \tag{51}$$

Equation (51) is satisfied, if

$$P=\begin{bmatrix} p_1 & p_2\\ p_2 & p_2\end{bmatrix},\ p_1>0,\quad p_2(p_1-p_2)>0 \tag{52}$$

4.3. *Example 3*

Consider the system

$$A=\begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix}, B=P=H=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} \tag{53}$$

and the state weight matrix

$$Q=kQ_0=k\begin{bmatrix} \frac{1}{4} & 0\\ 0 & -\frac{3}{4}\end{bmatrix} \tag{54}$$

For which real values of k can we establish a real symmetric solution to the ARE, Eqn. (5)? In Lemma 1 (5) has a real symmetric solution, if and only if,

$$Q-Q_0-X(\omega)=\begin{bmatrix} k\frac{1}{4}+\omega^2 & j\omega\\ -j\omega & -k\frac{3}{4}+1+\omega^2\end{bmatrix}\geqslant 0,\quad \forall\omega\in\mathbf{R}^1 \tag{55}$$

From Eqn. (55) we get the conditions

$$k\tfrac{1}{4}+\omega^2\geqslant 0,\quad \forall\omega\in\mathbf{R}^1 \tag{56}$$

$$\left(\omega^2-\frac{k}{4}\right)^2+\frac{k(1-k)}{4}\geqslant 0,\quad \forall\omega\in\mathbf{R}^1 \tag{57}$$

from (56) and (57) we find that a solution exists, if and only if,

$$0\leqslant k\leqslant 1 \tag{58}$$

5. Concluding remarks

For a class of linear time invariant systems, when $\dim(\boldsymbol{x})=\dim(\boldsymbol{u})=n$, we have shown that there is a finite upper bound on the imaginary value of the closed loop eigenvalues, no matter how the state weight matrix is chosen, provided there is a real symmetric solution of the ARE. When $\dim(\boldsymbol{x})\neq\dim(\boldsymbol{u})$, when $\det(H)\rightarrow 0$, the maximum imaginary value of the closed loop eigenvalues goes to infinity.

We have derived the maximum possible imaginary value $\beta_{\max}$, the corresponding worst case state weight matrix **Q** and the corresponding solution **R** to the algebraic Riccati equation. $\beta_{\max}$, **Q** and **R** are extreme solutions to the LQ-problem.

These solutions are uniquely determined by the system matrix **A**, the control input matrix **B** and the control input weight matrix **P**. These solutions give valuable insight into the LQ-optimal problem.

References

Di Ruscio, D. and Balchen, J. G. (1990). A Schur method for designing LQ-optimal systems with prescribed eigenvalues. *Modeling, Identification and Control*, **11**, 55–72.

Kučera, V. (1989). Algebraic Riccati equation: Symmetric and definite solutions. *Workshop on The Riccati equation in control systems, and signals. Como, Italy 1989*. With the participation of IEEE-Control Systems Society. Printed by Pitagora Editrice Bologna, Via del Legatore 3, Bologna.

Lancaster, P. and Rodman, L. (1980). Existence and uniqueness theorems for the algebraic Riccati equation. *Int. J. Control*, **32**, 285–309.

Laub, A. J. (1979) A Schur Method for Solving Algebraic Riccati Equations. *IEEE Trans. on Automatic Control*, **24**, 913–921.

Willems, J. C. (1971). Least Squares Stationary Optimal Control and the Algebraic Riccati Equation. *IEEE Trans. on Automatic Control*, **16**, 621–634.

Appendix A

In this appendix we will prove that the spectrum $\lambda(Z_0)$ is purely imaginary. Z_0 is defined in Equation (14), and can be written as

$$Z_0=-HZ_0^{\mathrm{T}}H^{-1} \tag{59}$$

The matrix **H** is assumed to be positive definite, then the Cholesky decomposition exists. We have

$$H=LL^{\mathrm{T}} \tag{60}$$

where $L\in\mathbf{R}^{n\times n}$ is a lower triangular matrix. Combining Eqn. (59) and (60) we get

$$L^{-1}Z_0L = -(L^{-1}Z_0L)^T \tag{61}$$

By similarity transformation we see that the spectrum

$$\lambda(Z_0) = \lambda(L^{-1}Z_0L) = -\lambda[(L^{-1}Z_0L)^T] \tag{62}$$

is purely imaginary because the matrix $L^{-1}Z_0L$ is skew symmetric. Note that the eigenvalues of a skew symmetric matrix are purely imaginary, and that the eigenvalues of a symmetric matrix are purely real.