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The stability of $m \times m$ multivariable process control systems†

JENS G. BALCHEN‡

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A technique is described for the investigation of conditions of stability of a $m \times m$ multivariable process controlled by m independent controllers in terms of the parameters of the m uncoupled control systems and the transfer functions of the cross-coupling elements. The method is an extension of a similar result for 2×2 systems (Balchen 1990). The analytical results are given convenient graphical representations by means of standard frequency response techniques yielding a simple, but powerful tool for synthesis of multiple controllers for multivariable processes

1. Introduction

The most common type of control system for a multivariable process is the multiple monovariable controller (diagonal controller matrix). The reason for this is its convenience in design and tuning and in most cases also acceptable performance. It is clear however, that a multivariable controller will in general yield a better performance, but has the inconveniences of higher complexity, less operator comprehension, more difficult tuning etc.

The pairing of variables in multivariable control is an important and interesting subject in itself which has been dealt with by many authors (McAvoy 1983, Balchen and Mumme 1988). In the following it shall be assumed that a proper pairing has been performed so that the analysis will be based upon a fixed structure of the model.

The problem at hand is to develop simple criteria for system stability of multivariable systems with multiple (diagonal) control in terms of the parameters of the individual uncoupled control systems and the properties of the cross-coupling elements. This problem can be solved in general terms, but the solution has little value with respect to understanding the system behaviour for systems that are of higher complexity than say 4×4 (4 inputs and 4 outputs). The knowledge gained by studying 2×2 and 3×3 systems however, is quite significant and is believed to settle most practical problems.

2. System analysis

The system shown in the block diagram of Fig. 1 is considered. $H_u(s)$ and $H_c(s)$ are the transfer matrices of the multivariable process and the multiple controllers

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‡ Division of Engineering Cybernetics, The Norwegian Institute of Technology, 7034 Trondheim.

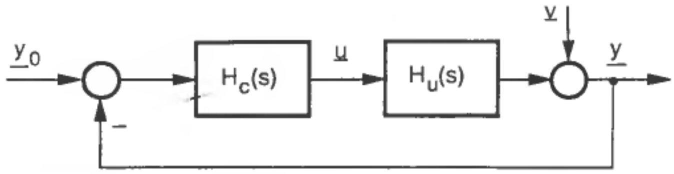

Figure 1. Block diagram of control system under consideration. $H_u(s)$: process transfer matrix. $H_c(s)$: controller transfer matrix.

respectively. The transfer from the vector of setpoints ($y_0(s)$) to the vector of outputs ($y(s)$) is given by

$$y(s)=(I+H_u(s)H_c(s))^{-1}H_u(s)H_c(s)y_0(s) \tag{1}$$

The stability of this system is determined by

$$\det(I+H_u(s)H_c(s)) \tag{2}$$

whose zeros are not allowed to be in the right half of the complex plane. Conditions securing that the zeros are in the left half of the complex plane are to be developed.

The matrix $H_u(s)=\{h_{ij}(s)\}$ is rewritten in the form

$$H_u(s)=H_u^d(s)+H_u^\Delta(s) \tag{3}$$

in which $H_u^d(s)$ constitutes the diagonal elements and $H_u^\Delta(s)$ the remaining elements of $H_u(s)$. In other words $H_u^\Delta(s)$ expresses the cross-coupling terms of the process transfer matrix. Since $H_c(s)=\text{diag}\{h_i(s)\}$ is a diagonal matrix (because we only consider multiple monovariable controllers), the product $H_u^d(s)H_c(s)$ will be a diagonal matrix.

Neglecting the argument s for simplicity, we get

$$\begin{aligned}(I+H_uH_c)&=(I+(H_u^d+H_u^\Delta)H_c)=(I+H_u^dH_c+H_u^\Delta H_c)\\&=(I+H_u^dH_c)(I+(I+H_u^dH_c)^{-1}H_u^\Delta H_c)\\&=(I+H_u^dH_c)(I+(I+H_u^dH_c)^{-1}H_u^dH_cH_c^{-1}(H_u^d)^{-1}H_u^\Delta H_c)\\&=T(I+M^dH_c^{-1}\tilde{H}_uH_c)\end{aligned} \tag{4}$$

in which

$$T=(I+H_u^dH_c)=\begin{bmatrix}T_1 & ..0 \\ \vdots & \\ & T_2 \\ & \vdots \\ 0 & T_m\end{bmatrix} \tag{5}$$

and

$$M^d=(I+H_u^dH_c)H_u^dH_c=\begin{bmatrix}M_1 & & ..0 \\ \vdots & M_2 & \vdots \\ & & \vdots \\ 0 & & M_m\end{bmatrix} \tag{6}$$

$$\text{and } \tilde{H}_u=(H_u^d)^{-1}H_u^\Delta \tag{7}$$

From (4) the determinant of (2) can be developed as

$$\det(I+H_uH_c)=\det T\cdot\det(I+M^dH_c^{-1}\tilde{H}_uH_c)$$
$$=\prod_{i=1}^{m} T_i\cdot\det(I+M^d\hat{H}) \tag{8}$$

where

$$\hat{H}=\{\hat{h}_{ij}\}=\left\{\frac{h_jh_{ij}}{h_ih_{ii}}\right\} \quad \text{for } i\neq j$$
$$=\{0\} \quad \text{for } i=j \tag{9}$$

The last term in (8) can be developed as

$$\det(I+M^d\hat{H})=\det\begin{bmatrix} 1 & M_1\hat{h}_{12} & M_1\hat{h}_{13} & M_1\hat{h}_{1m} \\ M_2\hat{h}_{21} & 1 & M_2\hat{h}_{23} & M_2\hat{h}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ M_m\hat{h}_{m1} & M_m\hat{h}_{m2} & M_m\hat{h}_{m3} & 1 \end{bmatrix} \tag{10}$$

The solution of (8) can now be illustrated first for a 3 × 3 system where

$$H_u=\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{11}$$

and

$$H_c=\begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} \tag{12}$$

We get

$$\det(I+H_uH_c)=T_1T_2T_3(1-YM_1M_2M_3) \tag{13}$$

where

$$T_i=1+h_{ii}h_i \tag{14}$$

and

$$M_i=\frac{h_{ii}h_i}{1+h_{ii}h_i} \tag{15}$$

and

$$Y=Y_{12}M_3^{-1}+Y_{13}M_2^{-1}+Y_{23}M_1^{-1}-P_{123} \tag{16}$$

with

$$Y_{ij}=\frac{h_{ij}h_{ji}}{h_{ii}h_{jj}} \tag{17}$$

and

$$P_{123}=\frac{h_{12}h_{23}h_{31}+h_{21}h_{32}h_{13}}{h_{11}h_{22}h_{33}} \tag{18}$$

The case of a 2×2 system comes out of (11) and (14) in that Y_{13}, Y_{23} and P_{123} are all equal to zero. So in that case we get

$$\det(I+H_uH_c)=T_1T_2(1-Y_{12}M_1M_2) \tag{19}$$

which is the result studied in Balchen (1990).

Similarly the solution for a 4×4 system can be found as

$$\det(I+H_uH_c)=T_1T_2T_3T_4(1-YM_1M_2M_3M_4) \tag{20}$$

where

$$\begin{aligned} Y=\ & Y_{12}M_3^{-1}M_4^{-4}+Y_{13}M_2^{-1}M_4^{-1}+Y_{23}M_1^{-1}M_4^{-1} \\ & +Y_{34}M_1^{-1}M_2^{-1}+Y_{24}M_1^{-1}M_3^{-1}+Y_{14}M_2^{-1}M_3^{-1} \\ & -P_{123}M_4^{-1}-P_{124}M_3^{-1}-P_{134}M_2^{-1}-P_{234}M_1^{-1} \\ & -P_{1234} \end{aligned} \tag{21}$$

and

$$Y_{ij}=\frac{h_{ij}h_{jj}}{h_{ii}h_{jj}} \tag{22}$$

$$P_{ijk}=\frac{h_{ik}h_{ji}h_{kj}+h_{ki}h_{ij}h_{jk}}{h_{ii}h_{jj}h_{kh}} \tag{23}$$

$$P_{1234}=\frac{Z_{1234}}{h_{11}h_{22}h_{33}h_{44}} \tag{24}$$

$$\begin{aligned} Z_{1234}=\ & (h_{12}h_{21}h_{34}-h_{14}h_{21}h_{32}-h_{12}h_{24}h_{31})h_{43} \\ & +(h_{13}h_{31}h_{24}-h_{14}h_{31}h_{23}-h_{13}h_{21}h_{34})h_{42} \\ & +(h_{14}h_{23}h_{32}-h_{13}h_{24}h_{32}-h_{12}h_{23}h_{34})h_{41} \end{aligned} \tag{25}$$

A general expression of conditions for asymptotic stability of $m\times m$ feedback control system will be that

$$\det(I+H_uH_e)=\prod_{i=1}^{m} T_i\cdot\left(1-Y\cdot\prod_{i=1}^{m} M_i\right) \tag{26}$$

does not have zeros in the right half plane.

Equation (26) expresses that for system stability each term must have its zeros in the left half plane. The zeros of the elements T_i express the stability of each individual control loop when the cross-couplings are not considered i.e. the other control loops are not closed.

Any $m\times m$ system will convert into a number of 2×2 systems if controllers in some of the loops are disconnected and put into 'manual' position. In this mode of operation it is still required that the system shall be stable. Therefore, in additiion to the requirement of (26) we will need to require that the expressions

$$(1-Y_{ij}M_iM_j) \qquad i=1,2\ldots m, \quad j=1,2\ldots m \tag{27}$$

shall not have zeros in the right half plane. This requirement is simpler to investigate since the expressions Y_{ij} are not dependent upon the controller settings.

Similarly a number of alternative combinations of 3 × 3 systems which have to be stable, will result from disconnecting controllers in $m \times m$ systems when $m \geqslant 4$ leading to the investigation of expressions of the type shown in (13). But since Y of (13) is dependent upon the controller settings, there is no particular advantage of investigating these requirements rather than the general requirement of (26). These questions will be illustrated in an example in a later paragraph.

Y of (26) may be determined from expressions like (16) or (21). Alternatively the frequency response $Y(j\omega)$ may be determined numerically as suggested by Di Ruscio (1990) by direct solution of (26) such that

$$Y(j\omega)=\frac{\prod_{i=1}^{m} T_i(j\omega)-\det(I+H_u(j\omega)H_c(j\omega))}{\prod_{i=1}^{m} h_{ii}(j\omega)h_i(j\omega)} \tag{28}$$

Conditions for the last term of (26) to have zeros only in the left half plane, must now be developed. Equivalently one may investigate the expression

$$\left(-\frac{1}{Y}+\prod_{i}^{m} M_i\right) \tag{29}$$

The case of 2 × 2 system is simpler than the 3 × 3 system because the function Y_{12} of (19) is only depending on process parameters (not the controller parameters) whereas in the 3 × 3 case the function Y of (16) is depending also on all the controller parameters through the factors M_i.

This leads to the proposal of an iterative procedure in which the function Y of (26) is calculated step by step on the basis of previously calculated functions M_i. Thus the procedure will be as follows

(1) Determine (tune) controllers h_i which are acceptable for the uncoupled case. Thereby the first iteration $M_i^{(1)}$ of the closed loop responses are determined.
(2) Calculate the first iteration $Y^{(1)}$ of (16), (21) etc.
(3) Determine a second iteration $M_i^{(2)}$ by tuning controllers h_i so that the zeros of (26) (e.g. (28)) are located in acceptable positions in the left half plane.
(4) Calculate the second iteration $Y^{(2)}$ and so on.

In most cases the above procedure will converge in few steps, particularly if the controllers are tuned one at a time.

3. Frequency response stability criterion

The conditions for the expression of (26) to have zeros only in the left half plane, can be determined using the Nyquist Stability Criterion. The frequency response locus of the function

$$Y(j\omega)\prod_{i=1}^{m} M_i(j\omega)$$

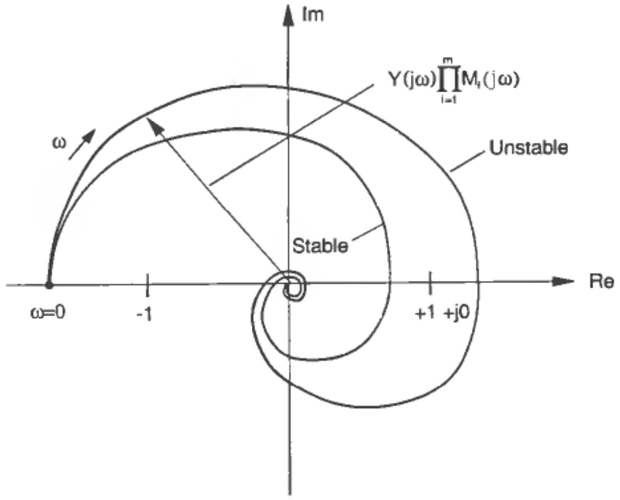

Figure 2. Complex plane plot of $Y(j\omega)\prod_{i=1}^{m} M_i(j\omega)$ showing a stable and an unstable system.

is plotted in the complex plane shown in Fig. 2. In accordance with the criterion, this locus shall not encircle the point $(+1+j\cdot 0)$ in the complex plane.

Alternatively the expressions of (29) could have been used and the behaviour of the locus of

$$\prod_{i=1}^{m} M_i(j\omega)$$

relative to the locus of

$$\frac{1}{Y(j\omega)}$$

could have been investigated. But this method would only have been advantageous in cases when $Y(j\omega)$ is constant with respect to changes in controller parameters as is the case for a 2×2 system.

A more convenient graphical representation than that of Fig. 2 is the Cartesian 'phase angle vs. dB amplitude' presentation as indicated in Fig. 3. Here it is seen that the locus of

$$Y(j\omega)\prod_{i=1}^{m} M_i(j\omega)$$

shall not pass above the point (0 dB, $-360°$).

In fact it is convenient to introduce stability margins in this graphical representation quite similar to that known from single loop systems using the Nichols chart.

4. Example

An example of a very simple 3×3 process control system will illustrate the method described above. The system is shown in Fig. 4. It consists of three control valves installed in a piping system in which three water streams are mixed. These streams have flows denoted $q_i (i=1,2\ldots)$, temperatures denoted θ_i and concentrations of a certain

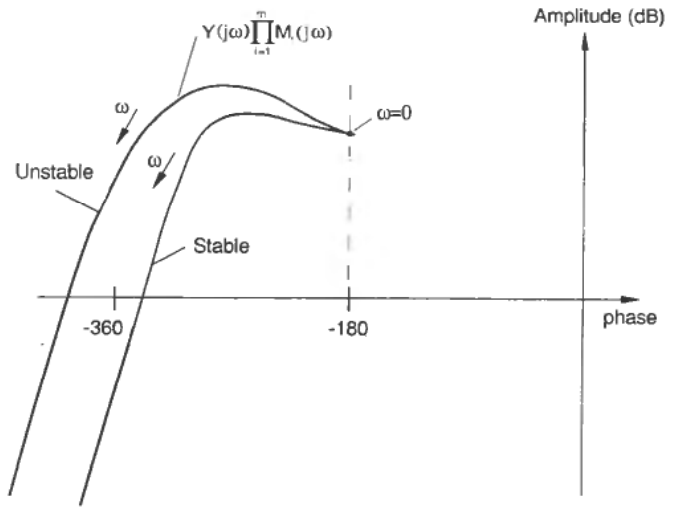

Figure 3. Same illustration in Fig. 2, but in Cartesian amplitude (dB) versus phase presentation.

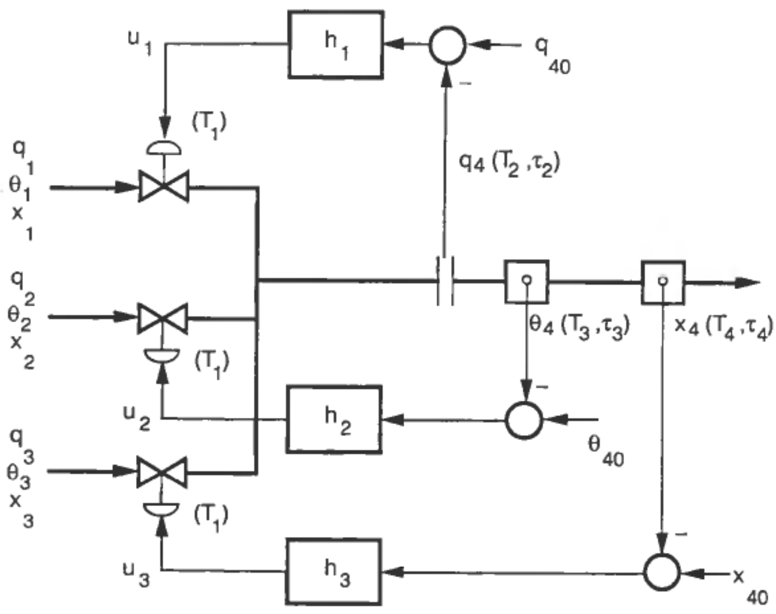

Figure 4. Simple 3 × 3 control system for mixing process.

agent (say colours) denoted x_i. The valves are assumed to be identical and linear and driven by membrane motors with identical time constants (T_1). The resulting flow after the mixing is denoted q_4. This is measured by a linear measuring device with a time constant T_2. Also associated with this measuring unit, there is a transportation delay denoted τ_2. The temperature of the mixture θ_4 is measured by a linear device with time constant T_3. Also there is a transportation delay from the valves to the temperature measurement denoted τ_3. The concentration of the specific agent in the mixture (x_4) is measured by a linear device with time constant T_4. The transportation delay from the valves to this measuring point is denoted τ_4.

Three monovariable controllers ($h_1(s)$, $h_2(s)$, and $h_3(s)$) are installed as shown in Fig. 4.

The following parameters characterize the system:

$$\theta_1=90\,(^\circ\mathrm{C})\quad x_1=0\quad q_{40}=\text{flow setpoint}=10\,(\mathrm{l/s})$$

$$\theta_2=10\,(^\circ\mathrm{C})\quad x_2=0\quad \theta_{40}=\text{temperature setpoint}=50\,(^\circ\mathrm{C})$$

$$\theta_3=20\,(^\circ\mathrm{C})\quad x_3=1\quad x_{40}=\text{concentration setpoint}=0{\cdot}5$$

$$T_1=1\,(\mathrm{s})$$

$$T_2=2\,(\mathrm{s})\quad \tau_2=1\,(\mathrm{s})$$

$$T_3=5\,(\mathrm{s})\quad \tau_3=10\,(\mathrm{s})$$

$$T_4=3\,(\mathrm{s})\quad \tau_4=15\,(\mathrm{s})$$

The process transfer functions of this system with the above data, will become

$$h_{11}(s)=\frac{e^{-s}}{(1+s)(1+2s)},\ h_{12}(s)=\frac{e^{-s}}{(1+s)(1+2s)},\ h_{13}(s)=\frac{e^{-s}}{(1+s)(1+2s)}$$

$$h_{21}(s)=\frac{4e^{-10s}}{(1+s)(1+5s)},\ h_{22}(s)=\frac{-4e^{-10s}}{(1+s)(1+5s)},\ h_{23}(s)=\frac{-3e^{-10s}}{(1+s)(1+5s)}$$

$$h_{31}(s)=\frac{-0{\cdot}05e^{-15s}}{(1+s)(1+3s)},\ h_{32}(s)=\frac{-0{\cdot}05e^{-15s}}{(1+s)(1+3s)},\ h_{33}=\frac{0{\cdot}05e^{-15s}}{(1+s)(1+3s)}$$

applying (17) and (18) this yields

$$Y_{12}=-1,\ Y_{13}=-1,\ Y_{23}=-0{\cdot}75,\ P_{123}=0{\cdot}25$$

If for simplicity it is assumed that each of the controllers h_i are ideal PID-controllers which are tuned so that the integral time is equal to the largest time constant and the derivative time is equal to the smallest time constant in each uncoupled loop and furthermore each loop is adjusted to a gain margin of 2 (6 dB), then it is easily shown that good approximations to the first iterations of the closed loop transfer functions will be

$$M_1(s)^{(1)}=\frac{1+0{\cdot}5s}{1+2\cdot 0{\cdot}485\cdot\dfrac{s}{1{\cdot}253}+\left(\dfrac{s}{1{\cdot}253}\right)^2}e^{-s}$$

$$M_2(s)^{(1)}=\frac{1+5s}{1+2\cdot 0{\cdot}485\cdot\dfrac{s}{0{\cdot}1253}+\left(\dfrac{s}{0{\cdot}1253}\right)^2}e^{-10s}$$

$$M_3(s)^{(1)}=\frac{1+7{\cdot}5s}{1+2\cdot 0{\cdot}485\cdot\dfrac{s}{0{\cdot}0835}+\left(\dfrac{s}{0{\cdot}0835}\right)^2}e^{-15s}$$

According to (16) we now get

$$Y^{(1)}=-[(M_3^{(1)})^{-1}+(M_2^{(1)})^{-1}+0{\cdot}75(M_1^{(1)})^{-1}+0{\cdot}25]$$

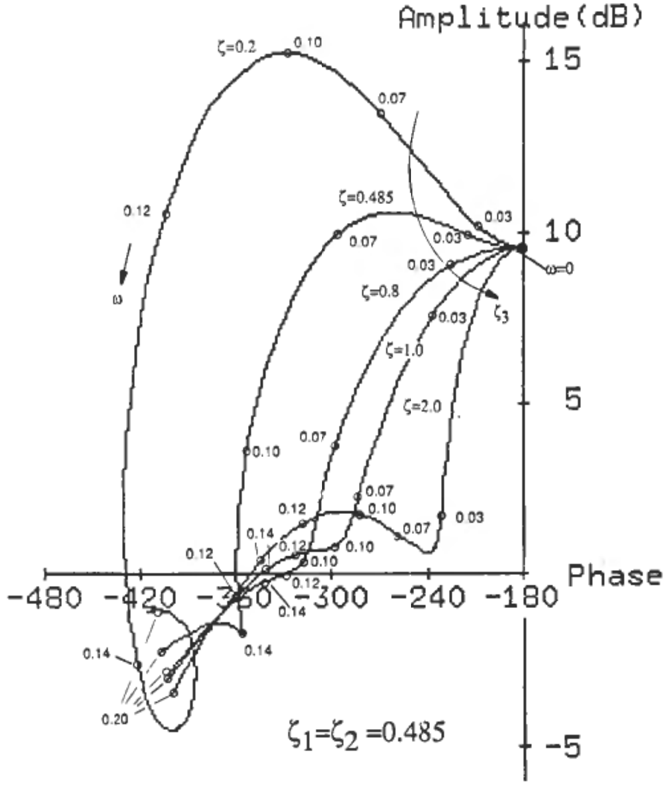

Figure 5. Amplitude versus phase diagram for nominal tuning of M_1 and M_2 with different tunings of M_3. The total system is either unstable or close to instability.

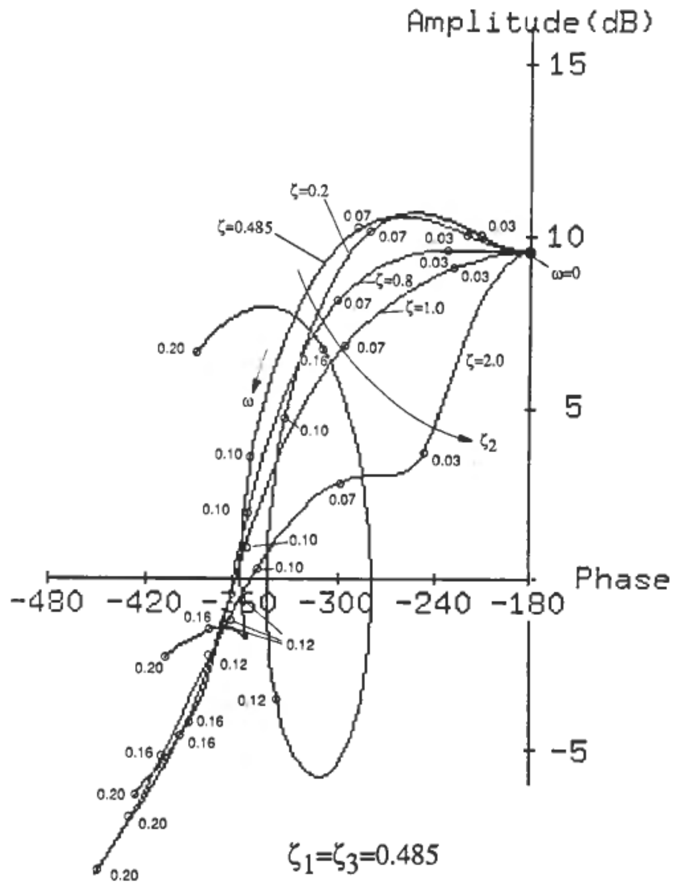

Figure 6. Amplitude versus phase diagram for nominal tuning of M_1 and M_3 with different tunings of M_2. The total system is either unstable or close to instability.

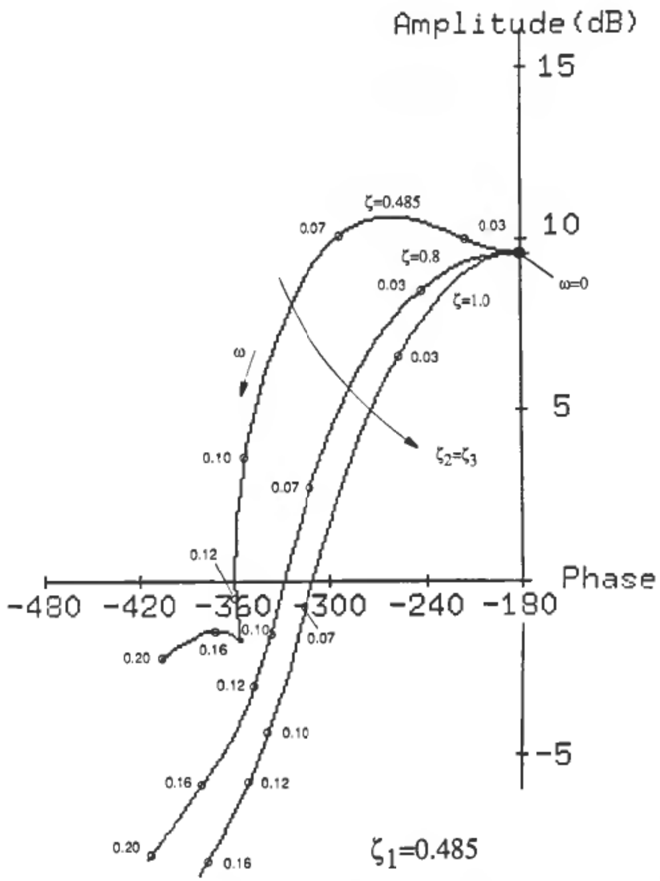

Figure 7. Amplitude versus phase diagram for nominal tuning of M_1 with different, but identical tunings of M_2 and M_3. The total system achieves acceptable stability margin when M_2 and M_3 have good damping.

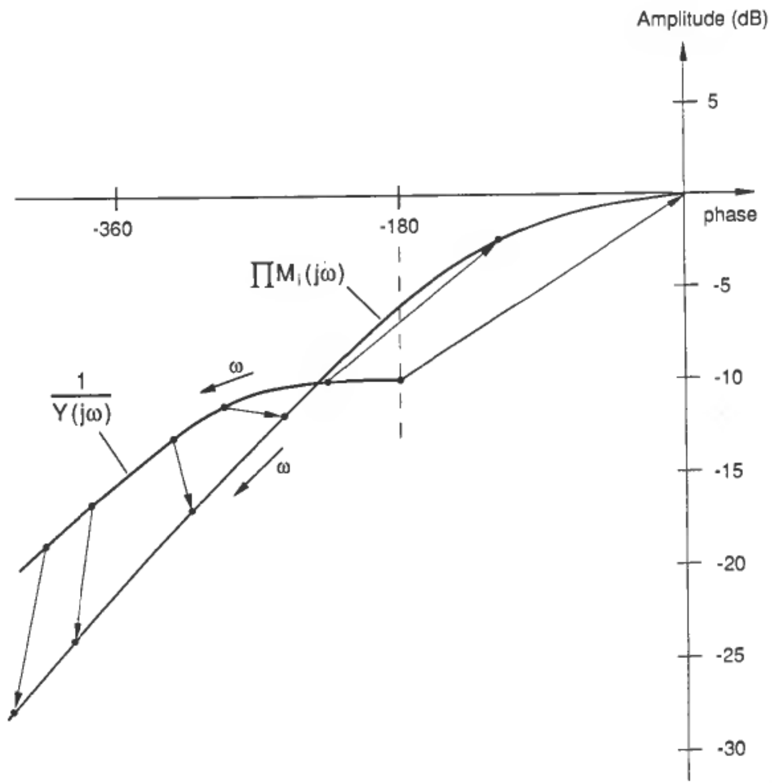

Figure 8. The behaviour of the locus $\prod M_i(j\omega)$ relative to the locus $\frac{1}{Y(j\omega)}$.

Now we want to investigate the behaviour of $Y(j\omega)M_1(j\omega)M_2(j\omega)M_3(j\omega)$. This is shown in Figs. 5–8 for different tunings of the different control loops. As can be seen, the loci all start at $(-180°, +9{\cdot}54\,\text{dB})$. In Fig. 5 the loops M_1 and M_2 (the flow and temperature loops) are tuned to the nominal dampings of $\zeta_1=\zeta_2=0{\cdot}485$ whereas the M_3-loop (concentration loop) is given different dampings (0·2–2·0). Clearly the system is unstable with $\zeta_3=0{\cdot}2$, on the limit of stability with $\zeta_3=0{\cdot}485$ and has very small margin of stability for $\zeta_3>0{\cdot}485$.

Since the flow loop M_1 is about one decade faster than the two other loops, it is clear that retuning of this loop will not influence very much the total behaviour. Therefore the next proposal will be to adjust the temperature loop while keeping the two other loops at their nominal tuning ($\zeta_1=\zeta_3=0{\cdot}485$). This is shown in Fig. 6. It is seen that for $\zeta_2=0{\cdot}2$ the system is unstable. For $\zeta_2\geqslant 0{\cdot}485$ the system is close to instability or has very small stability margin.

The last proposal is to adjust the damping of both the temperature loop and the concentration loop while keeping the flow loop at nominal tuning. This is shown in Fig. 7. Now it is clearly demonstrated that by introducing a damping of $\zeta_2=\zeta_3\geqslant 0{\cdot}8$ in these two loops the stability margin is greatly improved. With $\zeta_2=\zeta_3=1$, the gain margin becomes $\Delta K\approx 7\,\text{dB}$ and the phase margin $\Delta\psi\approx 50°$. A very similar result would have been obtained if the damping of the flow loop had been increased as well.

In some cases it may be advantageous to investigate the behaviour of the locus

$$\prod_{i=1}^{m} M_i(j\omega)$$

relative to the locus

$$\frac{1}{Y(j\omega)}.$$

Then the vector should be drawn for each frequency between these two loci as shown in Fig. 8. The system will be stable if this vector does not make a counter clock wise rotation.

5. Conclusions

The procedures developed above yield a very convenient insight into the behaviour of $m\times m$ multivariable control systems as experienced for instance in process control. Only seldom will it be necessary to investigate systems of higher dimension than $m=4$ for which analytical expressions of the appropriate transfer functions have been derived in (21)–(25). The technique allows for simple rules for tuning multiple controller systems. Such techniques have not been available previously.

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