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Convergence aspects of some robust estimators based upon prefiltering of the input/output data

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By prefiltering the input/output data and employing certain decentralized estimation techniques, it is possible to improve the robustness of some estimators significantly. Earlier papers on these techniques have been focused on local convergence properties of certain bootstrap estimators based upon these techniques. This paper is devoted to (1) global convergence properties, and (2) convergence rates when the underlying system is stiff.

1. Introduction

Parameter estimators based upon standard estimation techniques, viz. least squares (LS) methods, instrumental variable (IV) methods, etc. do occasionally have difficulties with systems that have a somewhat ill-conditioned nature, e.g. stiff systems. By prefiltering the input/output data and employing certain decentralized estimation techniques it is, however, possible to improve the robustness significantly. Previous papers on the methods presented in this paper have been focused on robustness properties and local convergence analysis, see Henriksen (1988, 1989) and Young *et al.* (1987). In this paper we shall focus our attention on two aspects: (1) global convergence analysis, and (2) convergence properties when the underlying system is stiff.

The paper is organized as follows. In § 2 we present a brief outline of the system and a resume of previous results concerning local convergence properties and robustness. § 3 is devoted to analysis of global convergence properties. Some results on convergence rates of the estimators when the underlying system is stiff are presented in § 4.

2. System description and previous results

We consider a system described by the linear discrete-time model

$$A(q^{-1})y_t = B(q^{-1})u_t + v_t \tag{1}$$

where y_t is the output at time t, u_t is the input, and v_t is the disturbance or residual. $\{v_t\}$ is assumed to be a zero-mean stochastic process with a rational nonsingular spectral density matrix. The processes $\{u_t\}$ and $\{v_t\}$ are assumed to be independent and the system is assumed to be asymptotically stable.

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In the previous papers by Henriksen (1988, 1989) it was originally assumed that the model (1) could be MIMO (multi-input, multi-output). The most neat and interesting results were obtained, however, by assuming the system to be SISO (single-input, single-output). We shall consequently, for the sake of simplicity, also make this assumption in the sequel.

The polynomials $A(q^{-1})$ and $B(q^{-1})$ are factored as, respectively,

$$A(q^{-1})=A_1(q^{-1})A_2(q^{-1}) \tag{2}$$

$$B(q^{-1})=B_1(q^{-1})B_2(q^{-1}) \tag{3}$$

where

$$A_1(q^{-1})=1+a_1^1q^{-1}+a_2^1q^{-2}+\ldots+a_{n_1}^1q^{-n_1} \tag{4}$$

$$A_2(q^{-1})=1+a_1^2q^{-1}+a_2^2q^{-2}+\ldots+a_{n_2}^2q^{-n_2} \tag{5}$$

$$B_1(q^{-1})=1+b_1^1q^{-1}+b_2^1q^{-2}+\ldots+b_{m_1}^1q^{-m_1} \tag{6}$$

$$B_2(q^{-1})=b_1^2q^{-1}+b_2^2q^{-2}+\ldots+b_{m_2}^2q^{-m_2} \tag{7}$$

and where $n_1+n_2=n$, the degree of $A(q^{-1})$, whereas $m_1+m_2=m$, the degree of $B(q^{-1})$.

Assuming the polynomials $A_2(q^{-1})$ and $B_2(q^{-1})$ to be known, we can define two new variables, viz.

$$w_t=A_2(q^{-1})y_t; \quad r_t=B_2(q^{-1})u_t \tag{8}$$

and (1) takes the form

$$A_1(q^{-1})w_t=B_1(q^{-1})r_t+v_t \tag{9}$$

which is a reduced model of the system.

Similarly, assuming $A_1(q^{-1})$ and $B_1(q^{-1})$ to be known, we can define

$$z_t=A_1(q^{-1})y_t; \quad s_t=B_1(q^{-1})u_t \tag{10}$$

which would lead to the reduced model

$$A_2(q^{-1})z_t=B_2(q^{-1})s_t+v_t \tag{11}$$

Equations (8)–(11) form the basis for the estimators considered in this paper. We can rewrite (9) and (11) as, respectively,

$$w_t=\psi_t^{\mathrm{T}}\beta+r_t+v_t \tag{12}$$

$$z_t=\phi_t^{\mathrm{T}}\theta+v_t \tag{13}$$

where

$$\psi_t=[-w_{t-1},\ldots,-w_{t-n_1},r_{t-1},\ldots,r_{t-m_1}]^{\mathrm{T}} \tag{14}$$

$$\beta=[a_1^1,\ldots,a_{n_1}^1,b_1^1,\ldots,b_{m_1}^1]^{\mathrm{T}} \tag{15}$$

$$\phi_t=[-z_{t-1},\ldots,-z_{t-n_2},s_{t-1},\ldots,s_{t-m_2}]^{\mathrm{T}} \tag{16}$$

$$\theta=[a_1^2,\ldots,a_{n_2}^2,b_1^2,\ldots,b_{m_2}^2]^{\mathrm{T}} \tag{17}$$

From (12) and (13) we can derive the following LS estimator

$$\hat{\beta}=\left[\frac{1}{N}\sum_{t=1}^{N}\psi_t\psi_t^{\mathrm{T}}\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\psi_t(w_t-r_t)\right] \tag{18}$$

$$\hat{\theta}=\left[\frac{1}{N}\sum_{t=1}^{N}\phi_t\phi_t^{\mathrm{T}}\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\phi_t z_t\right] \tag{19}$$

Since neither of the true values β^* and θ^* generally are known beforehand, the two estimators (18) and (19) will have to be employed in a bootstrap fashion by replacing w_t, r_t, z_t and s_t with, respectively,

$$\hat{w}_t=A_2(q^{-1},\hat{\theta})y_t;\quad \hat{r}_t=B_2(q^{-1},\hat{\theta})u_t \tag{20}$$

$$\hat{z}_t=A_1(q^{-1},\hat{\beta})y_t;\quad \hat{s}_t=B_1(q^{-1},\hat{\beta})u_t \tag{21}$$

For more details, see Henriksen (1988, 1989).

From (18) and (19) we can also immediately derive an IV estimator. It has the form

$$\hat{\beta}=\left[\frac{1}{N}\sum_{t=1}^{N}\tilde{\psi}_t\psi_t^{\mathrm{T}}\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\tilde{\psi}_t(w_t-r_t)\right] \tag{22}$$

$$\hat{\theta}=\left[\frac{1}{N}\sum_{t=1}^{N}\tilde{\phi}_t\phi_t^{\mathrm{T}}\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}\tilde{\phi}_t z_t\right] \tag{23}$$

where

$$\tilde{\psi}_t=[-\tilde{w}_{t-1},\ldots,-\tilde{w}_{t-n_1},r_{t-1},\ldots,r_{t-m1}]^{\mathrm{T}} \tag{24}$$

$$\tilde{w}_t=A_1^{-1}(q^{-1})B(q^{-1})u_t \tag{25}$$

$$\tilde{\phi}_t=[-\tilde{z}_{t-1},\ldots,-\tilde{z}_{t-n_2},s_{t-1},\ldots,s_{t-m_2}]^{\mathrm{T}} \tag{26}$$

$$\tilde{z}_t=A_2^{-1}(q^{-1})B(q^{-1})u_t \tag{27}$$

The convergence properties of the above estimators have been thoroughly investigated by Henriksen (1988, 1989) who uses the batch form above, and by Weyer (1988) who uses the recursive variants of the estimators. Provided certain standard consistency conditions are satisfied (e.g. (1) the polynomials $A(q^{-1})$ and $B(q^{-1})$ are coprime, (2) the input process $\{u_t\}$ is stationary, ergodic with respect to second-order moments, and persistently exciting of order m (LS) ($n+m$ in the noisefree case) or of order $n+m$ (IV), etc., see Henriksen 1989, or Söderström and Stoica 1983), we now summarize the local convergence properties in what follows.

After some elaborate computations it can be derived that local convergence (about the point (β^*,θ^*)) of the LS estimator can be determined from the eigenvalues of the matrix

$$F=[E\psi_t\psi_t^{\mathrm{T}}]^{-1}E\psi_t\phi_t^{\mathrm{T}}[E\phi_t\phi_t^{\mathrm{T}}]^{-1}E\phi_t\psi_t^{\mathrm{T}} \tag{28}$$

whereas local convergence of the IV variant can be determined from the eigenvalues of the matrix

$$\tilde{F}=[E\tilde{\psi}_t\tilde{\psi}_t^{\mathrm{T}}]^{-1}E\tilde{\psi}_t\tilde{\phi}_t^{\mathrm{T}}[E\tilde{\phi}_t\tilde{\phi}_t^{\mathrm{T}}]^{-1}E\tilde{\phi}_t\tilde{\psi}_t^{\mathrm{T}} \tag{29}$$

where E denotes the expectation operator.

Fact 1. If λ is an eigenvalue of F (or $\tilde{F}$), then λ is real and $0\leqslant\lambda\leqslant 1$. Moreover, $\lambda=1$ is an eigenvalue of F (or $\tilde{F}$) of multiplicity $k=i+j$ if and only if $A_1(q^{-1})$ and $A_2(q^{-1})$ have exactly i common zeros, and $B_1(q^{-1})$ and $B_2(q^{-1})$ have exactly j common zeros.

Fact 2. The above bootstrap estimators converge locally if both the polynomials $A_1(q^{-1})$ and $A_2(q^{-1})$ and the polynomials $B_1(q^{-1})$ and $B_2(q^{-1})$ are coprime.

Fact 2 holds for the IV variant provided $\{u_t\}$ and $\{v_t\}$ are uncorrelated, but a similar thing could occur if the two processes are correlated. Note that we in the above facts also have assumed $A(q^{-1})$ and $B(q^{-1})$ to be coprime.

Simulation experiments with these estimators clearly reveal that in cases with stiff systems where ordinary LS and IV estimators turn out to fail, the variants in this paper work quite well, see Young *et al.* (1987) and Henriksen (1988, 1989).

3. Global convergence analysis

For the purpose of doing the global convergence analysis, we decided to rewrite the estimators in the previous section in recursive form. This allows us to use the ODE-method developed by Ljung (1977) to carry out our analysis, see also Ljung and Söderström (1983).

The recursive form of the LS variant is given by the equations

$$P_t = P_{t-1} + \frac{1}{t}(\psi_t \psi_t^{\mathrm{T}} - P_{t-1}) \tag{30}$$

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \frac{1}{t} P_t^{-1} \psi_t (w_t - r_t - \psi_t^{\mathrm{T}} \hat{\beta}_{t-1}) \tag{31}$$

$$Q_t = Q_{t-1} + \frac{1}{t}(\phi_t \phi_t^{\mathrm{T}} - Q_{t-1}) \tag{32}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t} Q_t^{-1} \phi_t (z_t - \phi_t^{\mathrm{T}} \hat{\theta}_{t-1}) \tag{33}$$

where w_{t-l}, $l = 0, 1, \ldots, n_1$ can be approximated by

$$w_{t-l} = A_2(q^{-1}, \hat{\theta}_{t-l-1}) y_{t-l} \tag{34}$$

or by

$$w_{t-l} = A_2(q^{-1}, \hat{\theta}_{t-1}) y_{t-l} \tag{35}$$

Note that (35) requires that all w_{t-l}, $t = 1, \ldots, n_1$ are computed at each time instant. Similarly, z_{t-l}, $l = 0, 1, \ldots, n_2$ can be approximated by

$$z_{t-l} = A_1(q^{-1}, \hat{\beta}_{t-l}) y_{t-l} \tag{36}$$

or by

$$z_{t-l} = A_1(q^{-1}, \hat{\beta}_t) y_{t-l} \tag{37}$$

In Eqns. (36)–(37), note that the last update of $\hat{\beta}$, i.e., $\hat{\beta}_{t-l}$ or $\hat{\beta}_t$, is being used as soon as it is available (instead of using $\hat{\beta}_{t-l-1}$ or $\hat{\beta}_{t-1}$, respectively). If we do the same thing in the bootstrap estimator (18)–(19) this can be shown to double the rate of convergence locally, see Weyer (1988).

In a similar fashion, the first of the IV variants takes the recursive form

$$P_t = P_{t-1} + \frac{1}{t}(\tilde{\psi}_t \psi_t^{\mathrm{T}} - P_{t-1}) \tag{38}$$

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \frac{1}{t} P_t^{-1} \tilde{\psi}_t (w_t - r_t - \psi_t^{\mathrm{T}} \hat{\beta}_{t-1}) \tag{39}$$

$$Q_t = Q_{t-1} + \frac{1}{t}(\tilde{\phi}_t \phi_t^{\mathrm{T}} - Q_{t-1}) \tag{40}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t} Q_t^{-1} \tilde{\phi}_t (z_t - \phi_t^{\mathrm{T}} \hat{\theta}_{t-1}) \tag{41}$$

where the following approximations can be used:

$$\tilde{w}_{t-l} = A_1^{-1}(q^{-1}, \hat{\beta}_{t-l-1}) B(q^{-1}, \hat{\beta}_{t-l-1}, \hat{\theta}_{t-l-1}) u_{t-l} \tag{42}$$

or

$$\tilde{w}_{t-l} = A_1^{-1}(q^{-1}, \hat{\beta}_{t-1}) B(q^{-1}, \hat{\beta}_{t-1}, \hat{\theta}_{t-1}) u_{t-l} \tag{43}$$

$$\tilde{z}_{t-l} = A_2^{-1}(q^{-1}, \hat{\theta}_{t-l-1}) B(q^{-1}, \hat{\beta}_{t-l}, \hat{\theta}_{t-l-1}) u_{t-l} \tag{44}$$

or

$$\tilde{z}_{t-l} = A_2^{-1}(q^{-1}, \hat{\theta}_{t-1}) B(q^{-1}, \hat{\beta}_t, \hat{\theta}_{t-1}) u_{t-l} \tag{45}$$

We shall now carry out the global analysis for the recursive LS variant (30)–(33). Define

$$\kappa_t = [\beta_t^{\mathrm{T}}, \theta_t^{\mathrm{T}}]^{\mathrm{T}} \tag{46}$$

$$R_t = \begin{bmatrix} P_t & 0 \\ 0 & Q_t \end{bmatrix}; \quad \Pi_t = \begin{bmatrix} \psi_t & 0 \\ 0 & \phi_t \end{bmatrix} \tag{47}$$

$$p_t = [w_t - r_t, z_t]^{\mathrm{T}} \tag{48}$$

This allows us to rewrite (30)–(33) as

$$R_t = R_{t-1} + \frac{1}{t}(\Pi_t \Pi_t^{\mathrm{T}} - R_{t-1}) \tag{49}$$

$$\hat{\kappa}_t = \hat{\kappa}_{t-1} + \frac{1}{t} R_t^{-1} \Pi_t (p_t - \Pi_t^{\mathrm{T}} \hat{\kappa}_{t-1}) \tag{50}$$

We assume that a projection is used in the estimation algorithm to keep $\hat{\kappa}_t$ in D_M, where D_M is a compact subset of R^{n+m} such that $\kappa \in D_M$ implies that the system (1) is asymptotically stable. It is also assumed that we use a projection to keep R_t positive definite.

In accordance with Ljung (1977), see also Ljung and Söderström (1983), the convergence properties of (49)–(50) can be determined from the stability properties of the associated differential equations

$$\frac{d}{d\tau} \kappa(\tau) = R^{-1}(\tau) f(\kappa(\tau)) \tag{51}$$

$$\frac{d}{d\tau} R(\tau) = G(\kappa(\tau)) - R(\tau) \tag{52}$$

where

$$f(\kappa) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E(\Pi_t(\kappa) \varepsilon_t(\kappa)) = \bar{E}(\Pi_t(\kappa) \varepsilon_t(\kappa)) \tag{53}$$

$$G(\kappa) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E(\Pi_t(\kappa) \Pi_t^{\mathrm{T}}(\kappa)) = \bar{E}(\Pi_t(\kappa) \Pi_t^{\mathrm{T}}(\kappa)) \tag{54}$$

and where $\varepsilon_t(\kappa) = p_t(\kappa) - \Pi_t^{\mathrm{T}}(\kappa)\kappa$. Here $\varepsilon_t(\kappa)$, $p_t(\kappa)$ and $\Pi_t(\kappa)$ denote the stationary processes that would be obtained if, in the recursions generating ε_t, p_t and Π_t

respectively, the sequence of estimates κ_t is replaced by a constant parameter vector κ. We assume that these limits exist with probability 1 (w.p.1) for all $\kappa \in D_M$. $\bar{E}$ is defined as

$$\bar{E}h(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Eh(t)$$

Local analysis of (51)–(52) reveals exactly the same results as were obtained by Henriksen (1988, 1989) and which are summarized in Fact 1 and Fact 2. This also holds for the IV variant.

In order to investigate the global stability properties of (51)–(52) and hence the global convergence properties of (49)–(50), we introduce the Liapunov function

$$V(\kappa(\tau), R(\tau)) = \tfrac{1}{2}\bar{E}[A_1(q^{-1},\beta)A_2(q^{-1},\theta)y_t - B_1(q^{-1},\beta)B_2(q^{-1},\theta)u_t]^2 \tag{55}$$

We assume that the data sequence $\{y_t, u_t\}$ is such that this function will exist for all $\kappa \in D_M$.

We are now able to present the following result.

Theorem 1. Assume $A(q^{-1})$ and $B(q^{-1})$ to be coprime. Then the LS variant converges *globally*.

Proof. See the Appendix.

The proof of the above theorem reveals that the LS variant theoretically also converges when the coprimeness condition is not met. However, the only possible stationary point of (51)–(52), $f(\kappa(\tau)) = 0$, in this case turn out to be nonhyperbolic, i.e., there are local eigenvalues on the imaginary axis (a hyperbolic point is a stationary point where all local eigenvalues have real parts $\neq 0$). These eigenvalues will in our case be situated in the point $z = 0$, and their multiplicity μ will be exactly equal to the sum of the common zeros in $A_1(q^{-1})$ and $A_2(q^{-1})$, and in $B_1(q^{-1})$ and $B_2(q^{-1})$. Locally such a point will be unstable if $\mu > 1$, whereas it is stable, but not asymptotically stable, if $\mu = 1$. Anyway, the convergence property of the LS variant, which is reflected by (51)–(52), will in this case be very doubtful, and this has been fully confirmed by simulation experiments. Convergence has never been achieved when the coprimeness condition is not met.

On the other hand, if the coprimeness condition is met, then the local eigenvalues will all be strictly negative and real, which means that the stationary point will always be asymptotically stable. Convergence of the LS variant can in this case always be expected, but the convergence rate will be very low if, e.g., $A_1(q^{-1})$ and $A_2(q^{-1})$ have zeros which are close together (in relative terms).

Although the foregoing analysis does not exclude convergence when the coprimeness condition is not met (in fact, the analysis reveals that the LS variant does theoretically also converge in this case), there are several reasons why we should avoid that. First, the local convergence properties are very doubtful when this condition is not met. Second, the theorem presented in the next section shows that the convergence rate increases when the stiffness of the system increases, i.e., when the zeros of $A_1(q^{-1})$ and $A_2(q^{-1})$ are wide apart (relatively). Third, if the system is not stiff, there is actually no need to use this LS variant, since an ordinary LS estimator in this case will perform better.

What is said above about the LS variant does not necessarily apply to the IV variant, since we have not succeeded in proving global convergence of that.

4. Convergence rates when the underlying system is stiff

We are now going to analyse the convergence rates of our estimators when the underlying system is stiff. Since stiffness is a property which we connect with the eigenvalues or poles of the system, we shall for the sake of simplicity assume $B(q^{-1})=0$, i.e., the system is driven by noise only. This is equivalent to saying that $B(q^{-1})$ is known. We shall, for the sake of simplicity, limit ourselves to analysing the LS variant, which implies that $\{v_t\}$ has to be white. We assume $Ev_t^2=\sigma^2$. Furthermore, $A_1(q^{-1})$ and $A_2(q^{-1})$ are assumed to be coprime and the system is stable.

The terms $E\psi_t\psi_t^T$ etc. of the matrix F (Eqn. (28)) are now calculated directly using the power spectral density and the inverse z-transform. We obtain, for example,

$$\Phi_{\psi\psi}(z)=\frac{\sigma^2}{A_1^*(z)A_1(z)}\begin{bmatrix} z^{n_1-1} \\ \vdots \\ 1 \end{bmatrix}[z,\dots,z^{n_1}] \tag{56}$$

and

$$\begin{aligned} E\psi_t\psi_t^T &= \frac{1}{2\pi j}\oint_{|z|=1}\Phi_{\psi\psi}(z)\frac{dz}{z} \\ &= \sigma_2\sum_{k=1}^{n_1} r_{\psi\psi}(k)\begin{bmatrix}\gamma_k^{n_1-1}\\ \vdots \\ 1\end{bmatrix}[1,\dots,\gamma_k^{n_1-1}] \\ &= \sigma^2\begin{bmatrix}\gamma_1^{n_1-1} & \dots & \gamma_{n_1}^{n_1-1}\\ \vdots & & \vdots \\ 1 & \dots & 1\end{bmatrix}\begin{bmatrix} r_{\psi\psi}(1) & & \\ & \ddots & \\ & & r_{\psi\psi}(n_1)\end{bmatrix}\begin{bmatrix}1 & \dots & \gamma_1^{n_1-1}\\ \vdots & & \vdots \\ 1 & \dots & \gamma_{n_1}^{n_1-1}\end{bmatrix} \end{aligned} \tag{57}$$

where γ_k, $k=1,2,\dots,n_1$ are the zeros of $A_1^*(z)$, where

$$A_1^*(z)=z^{n_1}A_1(z^{-1}) \tag{58}$$

and where

$$r_{\psi\psi}(k)=\lim_{z\to\gamma_k}\frac{z-\gamma_k}{A_1^*(z)A_1(z)} \tag{59}$$

We have in the above also assumed the zeros of $A_1^*(z)$ to be distinct. Similar expressions are obtained for $E\psi_t\phi_t^T$, $E\phi_t\psi_t^T$ and $E\phi_t\phi_t^T$.

A stiff system is characterized by the fact that the poles can be grouped into two parts $\{\gamma_1,\gamma_2,\dots,\gamma_p\}$ and $\{\gamma_{p+1},\dots,\gamma_n\}$ where the first one represents the slow dynamics, whereas the second one represents the fast dynamics. We define

$$\rho_{\text{slow}}=\sup_{\gamma_i,\, i=1,\dots,p}\{1-|\gamma_i|\} \tag{60}$$

$$\rho_{\text{fast}}=\inf_{\gamma_i,\, i=p+1,\dots,n}\{1-|\gamma_i|\} \tag{61}$$

In a truly stiff system ρ_{slow} will be close to zero, whereas ρ_{fast} is not. We can therefore take the ratio $\rho_{\text{fast}}/\rho_{\text{slow}}$ as a measure of a system's stiffness by noting that the ratio increases as the stiffness increases.

We now assume that, say, $A_1(q^{-1})$ represents the slow dynamics of the underlying system, whereas $A_2(q^{-1})$ represents the fast dynamics. This enables us to present the following result.

Theorem 2. As the stiffness tends to infinity, all eigenvalues of F (or of $\tilde{F}$) tend to zero.

Proof. See the Appendix

Corollary 1. The convergence rate of the LS variant (and the two IV variants) increases as the stiffness increases. Moreover, as the stiffness tends to infinity, the local convergence (of the batch variants) will tend to become instantaneous.

5. Conclusion

We have considered the problem of estimating stiff systems by employing some specific decentralized estimators which are based upon filtering the input/output data in a certain manner. The estimators can be designed as LS or as IV variants. Local convergence properties have been thoroughly examined by Henriksen (1988, 1989) in previous papers. We have in this paper focused our attention on: (1) global convergence analysis, and (2) convergence rates for stiff systems. The global convergence analysis did in some way confirm what the local analysis gave, i.e., the estimators converge globally if the coprimeness condition is met. Moreover, the global analysis also showed that convergence theoretically would take place even when the coprimeness condition is not met. However, the convergence point (the true parameters (β^*, θ^*)) is in this case not hyperbolic, and very slow convergence, if any, can in this case be expected.

The analysis of the convergence rate showed what follows in order to ensure good convergence properties. For one thing, the slow and the fast modes of the system should be separated, i.e., put in the factors $A_1(q^{-1})$ and $A_2(q^{-1})$ respectively. For another, when the stiffness of the system increases, so does the convergence rate.

From the above it is apparent that some a priori knowledge of the system, e.g., that the system is stiff will be helpful in using these estimators for identification purposes. The estimators work better the stiffer the system is, as opposed to ordinary LS or IV variants, which often fail completely for stiff systems. On the other hand, if the system is not stiff, convergence of our estimators becomes slower, whereas ordinary LS and IV variants in this case normally would perform better.

Appendix

Proof of Theorem 1

We have the following expression

$$\frac{d}{d\tau}V(\kappa(\tau), R(\tau)) = \bar{E}(\hat{A}_1(q^{-1})\hat{A}_2(q^{-1})y_t - \hat{B}_1(q^{-1})\hat{B}_2(q^{-1})u_t)$$

$$\times \frac{d}{d\kappa}(\hat{A}_1(q^{-1})\hat{A}_2(q^{-1})y_t - \hat{B}_1(q^{-1})\hat{B}_2(q^{-1})u_t)R^{-1}(\tau)f(\kappa(\tau)) \qquad \text{(A 1)}$$

Careful evaluations lead to

$$\frac{d}{d\tau}V(\kappa(\tau), R(\tau)) = -f^{\mathrm{T}}(\kappa(\tau))R^{-1}(\tau)f(\kappa(\tau)) \qquad \text{(A 2)}$$

$$f(\kappa(\tau)) = \bar{E}(\hat{A}_1(q^{-1})\hat{A}_2(q^{-1})y_t - \hat{B}_1(q^{-1})\hat{B}_2(q^{-1})u_t) \begin{bmatrix} -\hat{A}_2(q^{-1})y_{t-1} \\ \vdots \\ -A_2(q^{-1})y_{t-n_1} \\ \hat{B}_2(q^{-1})u_{t-1} \\ \vdots \\ \hat{B}_2(q^{-1})u_{t-m_1} \\ -\hat{A}_1(q^{-1})y_{t-1} \\ \vdots \\ -\hat{A}_1(q^{-1})y_{t-n_2} \\ \hat{B}_1(q^{-1})u_{t-1} \\ \vdots \\ \hat{B}_1(q^{-1})u_{t-m_2} \end{bmatrix} \quad \text{(A 3)}$$

where $\hat{A}_1(q^{-1}) = A_1(q^{-1}, \beta(\tau))$ and so on. We now substitute for y_t, y_{t-1}, etc. in (A 3) from

$$y_t = \frac{B_1(q^{-1})B_2(q^{-1})}{A_1(q^{-1})A_2(q^{-1})}u_t + \frac{1}{A_1(q^{-1})A_2(q^{-1})}v_t \quad \text{(A 4)}$$

where $A_1(q^{-1}) = A_1(q^{-1}, \beta^*)$ and so on. Assuming $\{v_t\}$ to be white and independent of $\{u_t\}$, it is easily seen that we have to focus our attention on the factor in front of the vector in (A 3). Having substituted for y_t, this factor takes the form

$$\left(\frac{\hat{A}_1(q^{-1})\hat{A}_2(q^{-1})B_1(q^{-1})B_2(q^{-1})}{A_1(q^{-1})A_2(q^{-1})} - \hat{B}_1(q^{-1})\hat{B}_2(q^{-1})\right)u_t \quad \text{(A 5)}$$

and $f(\kappa) = 0$ if and only if

$$\hat{A}_1(q^{-1})\hat{A}_2(q^{-1})B_1(q^{-1})B_2(q^{-1}) = A_1(q^{-1})A_2(q^{-1})\hat{B}_1(q^{-1})\hat{B}_2(q^{-1}) \quad \text{(A 6)}$$

Apparently, $f(\kappa^*) = 0$ and there are no other values of κ which make $f(\kappa) = 0$ when $A(q^{-1})$ and $B(q^{-1})$ are coprime. Since $R(\tau)$ and hence $R^{-1}(\tau)$ are positive definite, we have that

$$\frac{d}{d\tau}V(\kappa(\tau), R(\tau)) < 0; \quad \kappa(\tau) \neq \kappa^* \quad \text{(A 7)}$$

Now, the ODE-method developed by Ljung (1977), see also Ljung and Söderström (1983), tells us that the estimate κ_t, will converge to κ^* or it will get stuck at the boundary of D_M. The estimate can get stuck at the boundary only if there is a trajectory of (51)–(52) that points out from D_M, i.e., if $V(\kappa, R)$ is decreasing as κ leaves D_M at some point. Now, at the boundary of the system's stability region D_s, $V(\kappa, R)$ tends to infinity. Hence, if the boundary of D_M is chosen close enough to the boundary of D_s, no trajectory will point out from D_M, and the algorithm will not converge to the boundary of D_M. This implies that the LS variant must converge w.p.1 to κ^*.

Notice that the system itself may be much more complex than the resulting model. But the model we end up with is the best approximation of the system in terms of a quadratic criterion. In the preceding calculation the true system hides behind the symbol $\bar{E}$.

Note that our analysis reveals that the coprimeness condition does not have to be met in order to have convergence. However, the point κ^* is in that case not hyperbolic, which means that real convergence of the LS estimator will be very doubtful or at least poor. This is in accordance with the local analysis which was done by Henriksen (1988, 1989).

Proof of Theorem 2

We assume all zeros of $A(q^{-1})$ to be distinct. This implies that the zeros of $A^*(z)=z^nA(z^{-1})$ are distinct. The following expressions for $E\psi_t\phi_t^T$, $E\phi_t\phi_t^T$ and $E\phi_t\psi_t^T$ are obtained, see Eqns. (56)–(59),

$$E\psi_t\phi_t^T=\sigma^2\sum_{k=1}^{n_1} r_{\psi\phi}(k)\begin{bmatrix}\gamma_k^{n_1-1}\\ \vdots\\ 1\end{bmatrix}[1,\ldots,\gamma_k^{n_2-1}] \tag{A 8}$$

$$E\phi_t\phi_t^T=\sigma^2\sum_{k=1}^{n_2} r_{\phi\phi}(k)\begin{bmatrix}\alpha_k^{n_2-1}\\ \vdots\\ 1\end{bmatrix}[1,\ldots,\alpha_{k_2}^{n}{}^{-1}] \tag{A 9}$$

$$E\phi_t\psi_t^T=\sigma^2\sum_{k=1}^{n_2} r_{\phi\psi}(k)\begin{bmatrix}\alpha_k^{n_2-1}\\ \vdots\\ 1\end{bmatrix}[1,\ldots,\alpha_k^{n_1-1}] \tag{A 10}$$

where $\alpha_1,\ldots,\alpha_{n_2}$ are the zeros of $A_2^*(z)=z^{n_2}A_2(z^{-1})$. Furthermore,

$$r_{\psi\phi}(k)=\lim_{z\to\gamma_k}\frac{z-\gamma_k}{A_1^*(z)A_2(z)} \tag{A 11}$$

$$r_{\phi\phi}(k)=\lim_{z\to\alpha_k}\frac{z-\alpha_k}{A_2^*(z)A_2(z)} \tag{A 12}$$

$$r_{\phi\psi}(k)=\lim_{z\to\alpha_k}\frac{z-\alpha_k}{A_2^*(z)A_1(z)} \tag{A 13}$$

Making use of (28), (57), and (A 8)–(A 10) we find the matrix F to be

$$\begin{bmatrix}1 & \cdots & \gamma_1^{n_1-1}\\ \vdots & & \vdots\\ 1 & \cdots & \gamma_{n_1}^{n_1-1}\end{bmatrix}^{-1}\begin{bmatrix}\frac{r_{\psi\phi}(1)}{r_{\psi\psi}(1)} & & \\ & \ddots & \\ & & \frac{r_{\psi\phi}(n_1)}{r_{\psi\psi}(n_1)}\end{bmatrix}\begin{bmatrix}1 & \cdots & \gamma_1^{n_2-1}\\ \vdots & & \vdots\\ 1 & \cdots & \gamma_{n_1}^{n_2-1}\end{bmatrix}$$

$$\times\begin{bmatrix}1 & \cdots & \alpha_1^{n_2-1}\\ \vdots & & \vdots\\ 1 & \cdots & \alpha_{n_2}^{n_2-1}\end{bmatrix}^{-1}\begin{bmatrix}\frac{r_{\phi\psi}(1)}{r_{\phi\phi}(1)} & & \\ & \ddots & \\ & & \frac{r_{\phi\psi}(n_2)}{r_{\phi\phi}(n_2)}\end{bmatrix}\begin{bmatrix}1 & \cdots & \alpha_1^{n_1-1}\\ \vdots & & \vdots\\ 1 & \cdots & \alpha_{n_2}^{n_1-1}\end{bmatrix} \tag{A 14}$$

We recognize matrices number 1 and number 4 to be inverse Vandermonde matrices. Since we have assumed the poles of the system to be distinct, this means that

$$\exists \varepsilon_0 > 0 \, \forall i \, \forall j \quad i \neq j | \gamma_i - \gamma_j | > \varepsilon_0 \wedge |\alpha_i - \alpha_j| > \varepsilon_0$$

This has to hold when the poles of the system tend to the unit circle, and the inverse Vandermonde matrices are therefore finite.

Notice that the zeros of $A_1(z)$ and $A_1^*(z)$ (and of $A_2(z)$ and $A_2^*(z)$) are each other's mirror images in the unit circle (i.e. if γ is a zero of $A_1^*(z)$ then $1/\bar{\gamma}$ is a zero of $A_1(z)$. $\bar{\gamma}$ denotes the complex conjugate of γ).

If we let a zero of $A_1^*(z)$, say γ_i, tend to the unit circle, it follows that $1/\bar{\gamma}_i$ also tends to the unit circle. If γ_i is real it must be the case that γ_i tends to $1/\gamma_i$, i.e., γ_i tends to a zero of $A_1(z)$. Likewise, if γ_i is complex, then γ_i tends to $1/\bar{\gamma}_i$ which is a zero of $A_1(z)$. The terms which appear in matrices number 2 and number 5 in (A 14) is by the above and Eqn. (59) given by

$$\frac{r_{\psi\phi}(i)}{r_{\psi\psi}(i)} = \lim_{z \to \gamma_i} \frac{A_1(z)}{A_2(z)} = \frac{\prod_{k=1}^{n_2} \alpha_k \prod_{k=1}^{n_1} (\gamma_i - \bar{\gamma}_k^{-1})}{\prod_{k=1}^{n_1} \gamma_k \prod_{k=1}^{n_2} (\gamma_i - \bar{\alpha}_k^{-1})} \tag{A 15}$$

$$\frac{r_{\phi\psi}(i)}{r_{\phi\phi}(i)} = \lim_{z \to \alpha_i} \frac{A_2(z)}{A_1(z)} = \frac{\prod_{k=1}^{n_1} \gamma_k \prod_{k=1}^{n_2} (\alpha_i - \bar{\alpha}_k^{-1})}{\prod_{k=1}^{n2} \alpha_k \prod_{k=1}^{n_1} (\alpha_i - \bar{\gamma}_k^{-1})} \tag{A 16}$$

Now, letting all zeros of $A_1^*(z)$ tend to the unit circle, which implies that the stiffness tends to infinity, we find that every element of matrix number 2 in (A 14) tends to zero, i.e., the matrix tends to zero. Therefore, the matrix F will also have to tend to the zero matrix. By Gershgorin's theorem, if λ is an eigenvalue of $F = (f_{i,j})$, then for some j $(1 \leqslant j \leqslant n)$

$$|f_{j,j} - \lambda| \leqslant \sum_{k=1, k \neq j}^{n} |f_{j,k}| \tag{A 17}$$

Since all elements of F tend to zero, it immediately follows that λ tends to zero, i.e., all eigenvalues of F tend to zero.

The above argument was made with the implicit assumption that the zeros of $A_2^*(z)$ remain fixed. This is sufficient to ensure that the eigenvalues tend to zero. It should, however, from (A 15)–(A 16) be clear that we also can allow the zeros of $A_2^*(z)$ to tend to the unit circle. Notice however that we have assumed all zeros of $A^*(z)$ distinct. This means that there is a positive distance, $\delta > 0$, between these zeros, and this must also be the case in the limit.

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