This approximation will be referred to as 'approximation A_n' where n is the order. It is usually assumed that the accuracy of the approximation is improved by increasing the number of terms in (2). However as will be shown below, this is not necessarily so.

Another common approximation has the form

$$e^{-\tau s} \approx \frac{\left(1-\frac{\tau}{2n}s\right)^n}{\left(1+\frac{\tau}{2n}s\right)}$$

$$= \frac{1-n\left(\frac{\tau}{2n}s\right)+\frac{n(n-1)}{2!}\left(\frac{\tau}{2n}s\right)^2-\frac{n(n-1)(n-2)}{3!}\left(\frac{\tau}{2n}s\right)^3+\dots}{1+n\left(\frac{\tau}{2n}s\right)+\frac{n(n-1)}{2!}\left(\frac{\tau}{2n}s\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{\tau}{2n}s\right)^3+\dots}$$

$$= \frac{1-b_1 s+b_2 s^2-b_3 s^3+\dots}{1+b_1 s+b_2 s^2+b_3 s^3+\dots}$$

$$= \frac{1-\frac{\tau}{2}s+\frac{\tau^2}{12}s^2-\frac{\tau^3}{216}s^3}{1+\frac{\tau}{2}s+\frac{\tau^2}{12}s^2+\frac{\tau^3}{216}s^3} \quad (\text{for } n=3) \tag{3}$$

This is 'approximation B_n'.

The problem with these approximations is that they do not seem to be the best possible when the power series are truncated to the second or third orders for instance.

The purpose of the present investigation is to find an approximation of the form

$$e^{-\tau s} = \frac{1-c_1 s+c_2 s^2-c_3 s^3+\dots}{1+c_1 s+c_2 s^2+c_3 s^3+\dots} \tag{4}$$

in which c_1, c_2, $c_3 \dots$ are to be determined so that the approximation meets certain specifications. This can be done in a number of different ways where the success is dependent upon the application of the results. We shall assume that the result will be used primarily for stability analysis of feedback systems. Therefore it makes sense to look at the frequency response of the transfer function and its approximation, namely

$$e^{-j\tau\omega} \approx \frac{1-jc_1\omega-c_2\omega^2+jc_3\omega^3-\dots}{1+jc_1\omega-c_2\omega^2-jc_3\omega^3+\dots}$$

$$= \frac{1-c_2\omega^2-j(c_1\omega-c_3\omega^3)-\dots}{1-c_3\omega^2+j(c_1\omega-c_3\omega^3)+\dots} \tag{5}$$

It is observed that the real and imaginary terms appearing in the numerator and the denominator of (5) have the same magnitude. This is a property of the balanced approximation.

3. Approximating the frequency response

Since the magnitude of the complex vector of (5) is unity for all frequencies ω, the approximation must be determined from the phase angle.

The phase angle of the exact frequency response of a delay is

$$\angle e^{-j\tau\omega} = -\tau\omega \tag{6}$$

whereas the approximation of (5) yields the phase angle

$$\angle e^{-j\tau\omega} \approx -2 \arctan\left(\frac{c_1\omega - c_3\omega^3}{1 - c_2\omega^2}\right) \qquad \text{for } n=3 \tag{7}$$

One way of determining the unknown coefficients of (5) would be to minimize the integral of the squared error between (6) and (7) over a certain frequency range. We shall choose to determine the unknown coefficients by prescribing that the approximation (7) is to fit the exact expression (6) at certain angles namely $-\pi/2$, $-\pi$, $-3\pi/2$. Any other set of angles could have been chosen but these are advantageous because they play important roles in the analysis of many control systems. This will be referred to as 'approximation C_n'.

When the approximation of (5) is of the first order ($c_2 = c_3 = 0$), the maximal negative phase angle will be $-\pi$ and we are free to choose one phase angle where the approximation is to fit the exact value. Choosing this to be $-\pi/2$, we obtain the coefficient of the first order approximation in (7) through

$$-2 \arctan c_1\omega = -\tau\omega = -\frac{\pi}{2} \tag{8}$$

giving

$$\omega_1 = \frac{\pi}{2\tau} \tag{9}$$

and

$$-\arctan c_1\omega = -\frac{\pi}{4}$$

This leads to

$$c_1\omega = 1$$

or

$$c_1 = \frac{2}{\pi}\tau \tag{10}$$

Thus the first order approximation which has the property of giving the correct phase angle at $-\pi/2$ will be

$$C_1\colon\ e^{-\tau s} \approx \frac{1 - \frac{2}{\pi}\tau s}{1 + \frac{2}{\pi}\tau s} \tag{11}$$

This can be compared with the approximations A_1 of (2) and B_1 of (3) and it is seen that the coefficient (c_1) has changed from $0{\cdot}5\,\tau$ to $0{\cdot}637\,\tau$.

Evaluating C_2 (the second order C approximation), we can make (6) and (7) equal at two frequencies, namely $\omega_1 = \pi/2\tau$ and $\omega_2 = \pi/2$. At these frequencies the phase angles are $-\pi/2$ and $-\pi$ respectively.

Applying (7) yields

$$\frac{c_1\omega_1}{1-c_2\omega_1^2}=\frac{\pi}{4}=1 \tag{12}$$

$$\frac{c_1\omega_2}{1-c_2\omega_2^2}=\tan\frac{\pi}{2}=\infty \tag{13}$$

(13) yields

$$c_2=\left(\frac{1}{\omega_2}\right)^2=\left(\frac{\tau}{\pi}\right)^2$$

which when applied in (12) leads to

$$c_1=\frac{1}{\omega_1}-c_2\omega_1=\frac{3\tau}{2\pi} \tag{15}$$

Thus the second order approximation with the property that the phase angle is exact at $-\pi/2$ and $-\pi$ will be

$$C_2\colon e^{-\tau s}\approx\frac{1-\frac{3\tau}{2\pi}s+\left(\frac{\tau}{\pi}\right)^2 s^2}{1+\frac{3\tau}{2\pi}s+\left(\frac{\tau}{\pi}\right)^2 s^2} \tag{16}$$

Comparing the result of (16) with the approximations A_2 of (2) and B_2 of (3) we see that the first order coefficient (c_1) has changed from 0·5 τ to 0·477 τ and that the second order coefficient (c_2) has changed from 0·125 τ^2 in approximation A_2 and 0·0833 τ^2 in B_2 to 0·101 τ^2 in A_2. In other words, the second order coefficient (c_2) in C_2 lies between those of A_2 and B_2.

Proceeding to the third order approximation, we find from (6) and (7)

$$\frac{c_1\omega_1-c_3\omega_1^3}{1-c_2\omega_1^2}=\tan\frac{\pi}{4}=1 \tag{17}$$

$$\frac{c_1\omega_2-c_3\omega_2^3}{1-c_2\omega_2^2}=\tan\frac{\pi}{2}=\infty \tag{18}$$

$$\frac{c_1\omega_3-c_3\omega_3^3}{1-c_2\omega_3^2}=\tan\frac{3\pi}{4}=-1 \tag{19}$$

where

$$\omega_1=\frac{\pi}{2\tau},\quad \omega_2=\frac{\pi}{\tau}\quad\text{and}\quad \omega_3=\frac{3\pi}{2\tau}.$$

(18) yields

$$c_2=\left(\frac{\tau}{\pi}\right)^2=\frac{\tau^2}{9{\cdot}87} \tag{20}$$

which when applied to (17) and (19) leads to

$$c_1=\frac{\tau}{\frac{12}{19}\pi}=\frac{\tau}{1{\cdot}984} \tag{21}$$

and

$$c_3=\frac{1}{3}\left(\frac{\tau}{\pi}\right)^3=\frac{\tau^2}{93{\cdot}02} \tag{22}$$

Consequently, the third order approximation which offers exact phase shift at $-\pi/2$, $-\pi$ and $-3\pi/2$ will be

$$C_3\colon e^{-\tau s}\approx\frac{1-\dfrac{\tau}{\frac{12}{19}\pi}s+\dfrac{\tau^2}{\pi^2}s^2-\dfrac{\tau^3}{3\pi^3}s^3}{1+\dfrac{\tau}{\frac{12}{19}\pi}s+\dfrac{\tau^2}{\pi^2}s+\dfrac{\tau^3}{3\pi^3}s^3}$$

$$=\frac{1-\dfrac{\tau}{1{\cdot}984}s+\dfrac{\tau^2}{9{\cdot}87}s^2-\dfrac{\tau^3}{93{\cdot}02}s^3}{1+\dfrac{\tau}{1{\cdot}984}s+\dfrac{\tau^2}{9{\cdot}87}s^2+\dfrac{\tau^3}{93{\cdot}02}s^3} \tag{23}$$

The latter expression can be compared with (2) and (3) and it is seen that the second and third order coefficients in particular are quite different.

A general formula for determining coefficients c_1, c_2, $c_3\ldots$ can be developed. Expressing these unknown coefficients as a vector

$$\boldsymbol{c}=[c_1c_2c_3\ldots c_n]^{\mathrm{T}}$$

the conditions of (17)–(19) can in general be expressed as

$$\boldsymbol{\Omega}_n\boldsymbol{c}=\boldsymbol{p} \tag{24}$$

where $\boldsymbol{\Omega}_n$ for the case of $n=5$ has the form

$$\boldsymbol{\Omega}_5=\begin{bmatrix}\omega_1 & \omega_1^2 & -\omega_1^3 & -\omega_1^4 & -\omega_1^5\\ 0 & \omega_2^2 & 0 & -\omega_2^4 & 0\\ \omega_3 & -\omega_3^2 & -\omega_3^3 & \omega_3^4 & -\omega_3^5\\ \omega_4 & 0 & -\omega_4^3 & 0 & -\omega_4^5\\ \omega_5 & \omega_5^2 & -\omega_5^3 & -\omega_5^4 & -\omega_5^5\end{bmatrix} \tag{25}$$

where

$$\omega_n=n\frac{\pi}{2} \tag{26}$$

Furthermore $\boldsymbol{p}$ for the case $n=5$ in (24) is

$$\boldsymbol{p}=[1\quad 1\quad -1\quad 0\quad 1]^{\mathrm{T}}$$

The solution of (24) is

$$\boldsymbol{c}=\boldsymbol{\Omega}_n^{-1}\boldsymbol{p} \tag{27}$$

The results of (11), (16) and (23) are easily verified using (27) with $n=1$, 2 and 3 respectively.

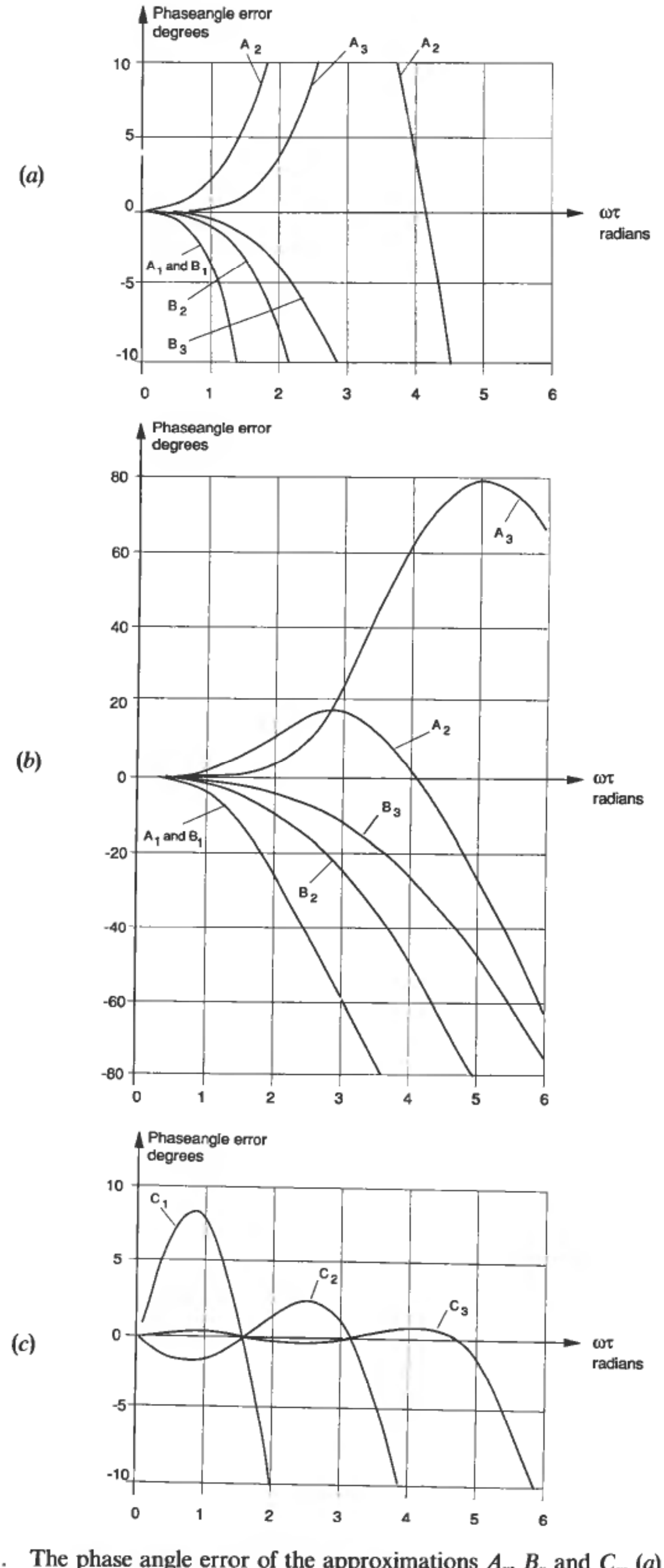


Figure 1. The phase angle error of the approximations A_n, B_n and C_n. (*a*) A_n and B_n at low frequencies. (*b*) A_n and B_n at higher frequencies. (*c*) C_n.

The errors associated with each of the three approximations have been investigated and are shown graphically in Fig. 1. It is clearly demonstrated that the new type of approximation (C_n) is superior to that of (A_n) and (B_n). The maximal errors indicated in the figure are quite small.

In some applications it is interesting to compare the pole-zero-plot of each of the approximations. Such plots are shown in Fig. 2 for the three different approximations studied above. It is observed that the A_5 and A_6 approximations have two poles in the right half of the s-plane. This means that the approximation is an unstable system and cannot be used alone. But these poles will be forced towards the left in a feedback system and thereby still make an acceptable approximation.

A third way of comparing the different approximations, is by simulating their time responses. This can be done by deriving the equivalent state space models for each of the rational transfer functions.

Two different input time functions have been applied for test purposes. The first input function is a step function (Laplace transform $u_1(s)=1/s$) and the second is a 'soft' step function (Laplace transform $u_2(s)=1/s(1+\tau s)$, i.e. with a time constant equal to the time delay τ.

The results are shown in Figs. 3–7. Figure 3 shows the step response of approximation A_n for $n=1,2\ldots 5$. As can be seen, the approximation A_5 is unstable as indicated above. It can hardly be said that the approximation improves with increasing n. Figure 4 shows the responses to a 'soft' step of approximations A_n. These appear to be much more successful than the pure step response. The reason is that a soft step contains less energy at high frequencies.

Figure 5 shows the step response and Fig. 6 the 'soft' step response of approximations B_n. In this case it is fair to say that the success of the approximation improves with increasing n. This is particularly clear for the 'soft' step response in Fig. 6.

Figures 7 and 8 show the responses of approximations C_n to step inputs and 'soft' step inputs respectively. Again it is observed that the approximation improves with increasing n.

Figures 9 (a) and 9 (b) compares the 'soft' step response of the three approximations with $n=2$ and $n=3$ respectively. It is evident that approximation C comes out best even though the differences between the three are not very significant.

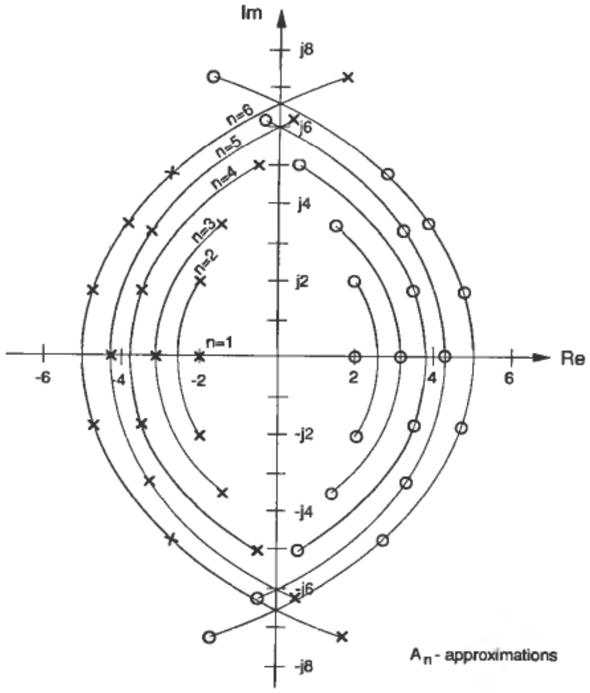
A conclusion to be drawn from the above investigations is that the approximation C_3 (as given in (23)), is very successful and should be used where high accuracy is required. Approximation C_2 is the seond best and will probably be acceptable in most cases even for stability investigations.

4. Applications

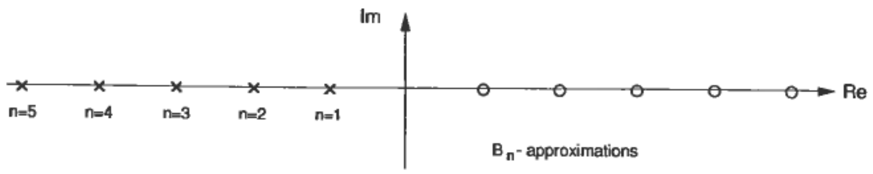
The rational function approximation to the transfer function of delay can be used for a number of purposes (Balchen 1977). It is immediately observed with algebraic expressions of conditions for stability of feedback systems of the form shown in Fig. 10 that using the approximations C_1, C_2 and C_3 the result will be exact if the phase angle of $h(j\omega)$ is $-\pi/2$. Such is the case if

$$h(s)=\frac{K}{s} \tag{28}$$

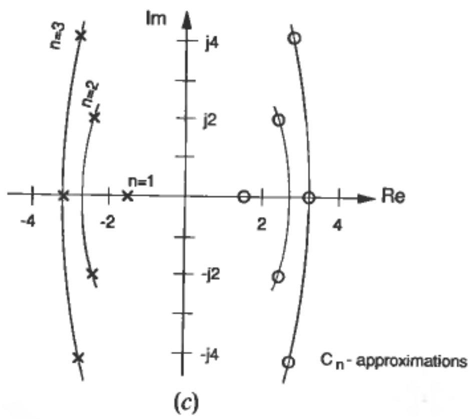
and also for other transfer functions under certain conditions. The approximations of C_2 and C_3 will yield the exact answer if the phase angle of $h(j\omega)$ is 0 which will occur if



(*a*)



(*b*)



(*c*)

Figure 2. Pole-zero-configuration of different approximations. (*a*) A_n-approximations. (*b*) B_n-approximations. (*c*) C_n-approximations.

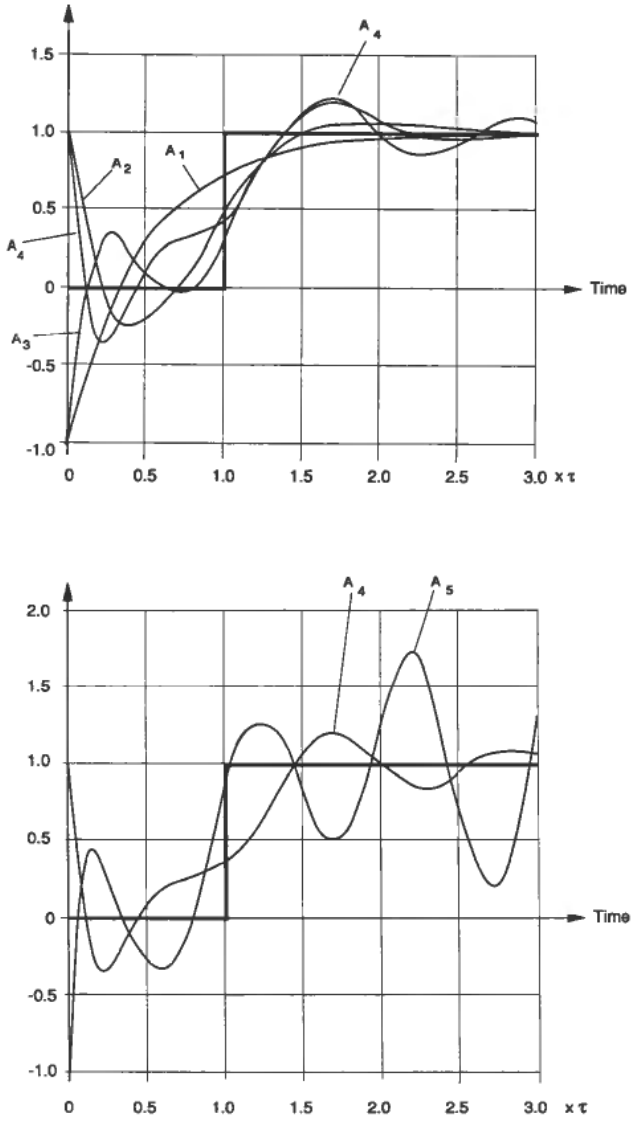


Figure 3. Step response of A_n-approximations.

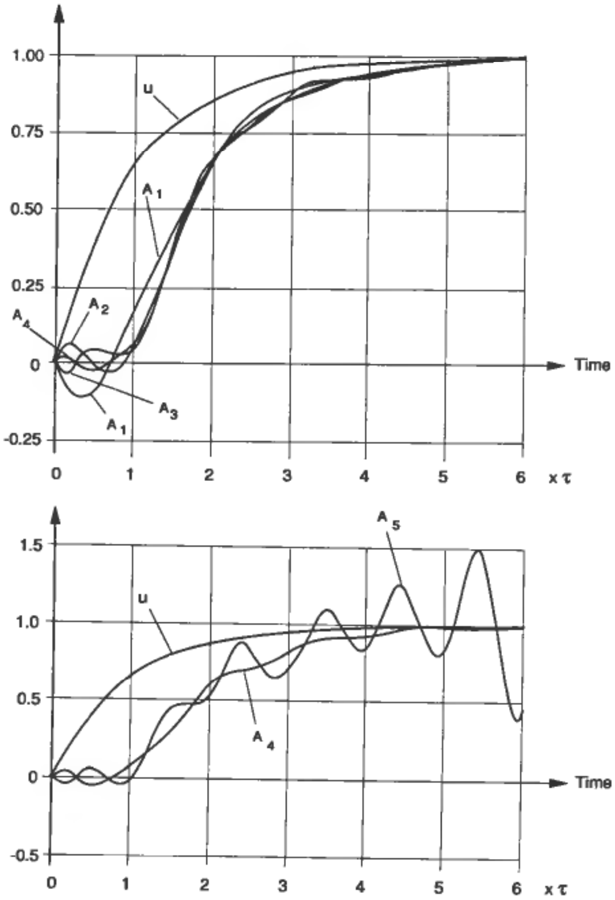


Figure 4. 'Soft' step response of A_n-approximations.

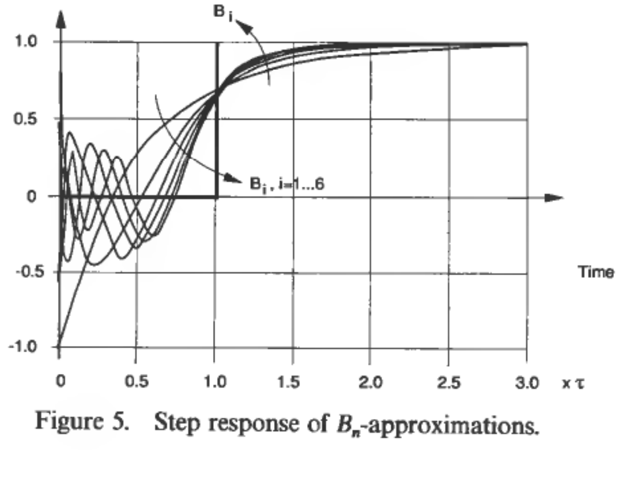


Figure 5. Step response of B_n-approximations.

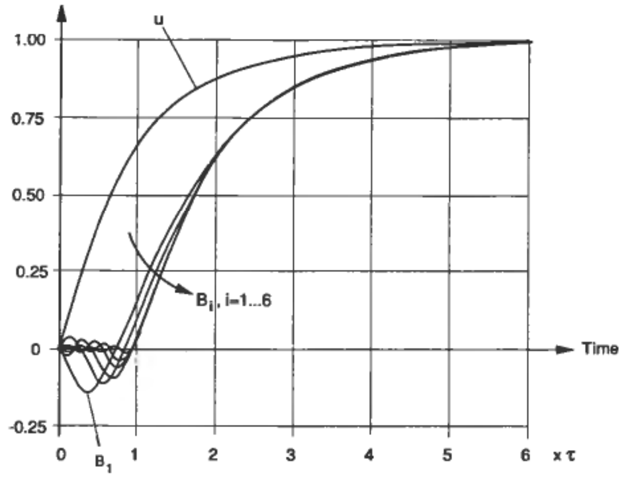


Figure 6. 'Soft' step response of B_n-approximations.

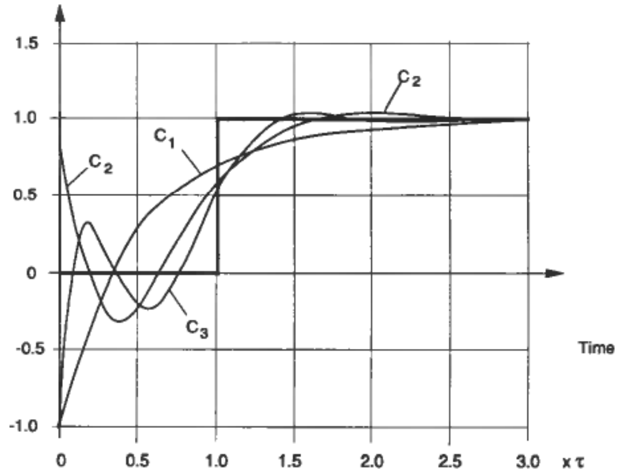


Figure 7. Step response of C_n-approximations.

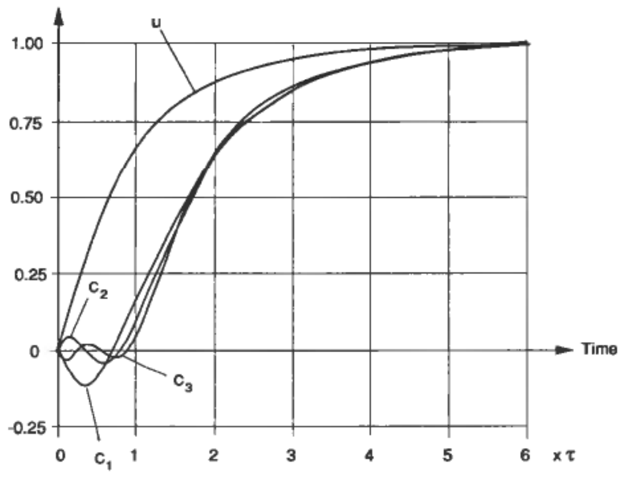
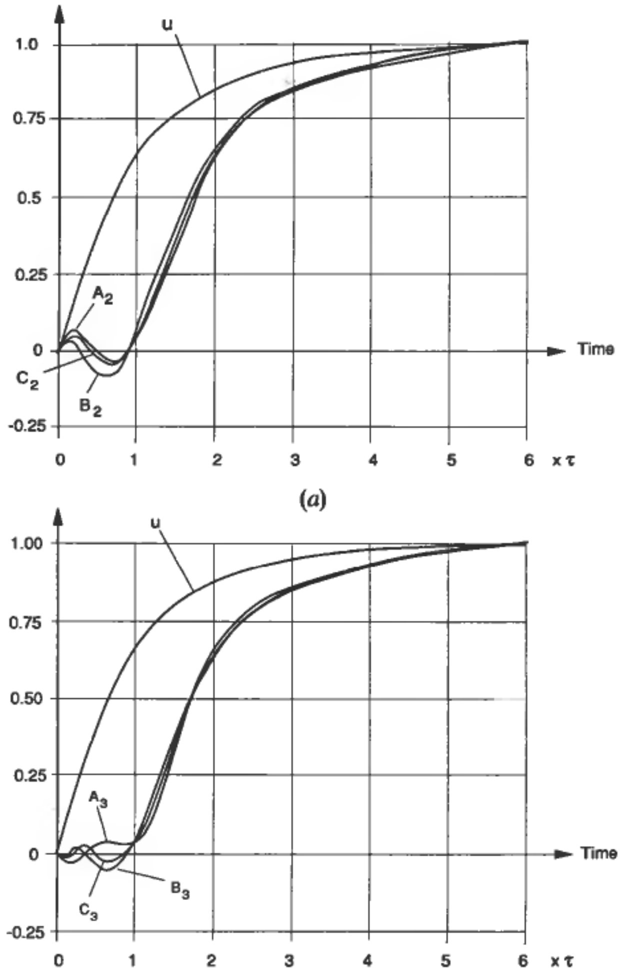


Figure 8. 'Soft' step response of C_n-approximations.



(*b*)

Figure 9. Comparison of 'soft' step responses. (*a*) A_2, B_2, C_2-approximations. (*b*) A_3, B_3, C_3-approximations.

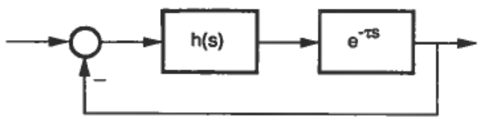


Figure 10. Feedback control system involving transport delay.

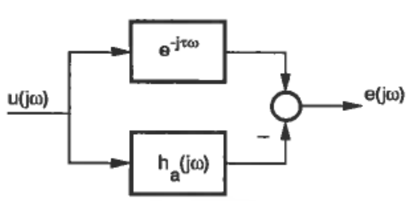


Figure 11. Generating the error between the transport delay and its approximation in the frequency domain.

$h(s) = K$ and for other transfer functions under certain conditions. Since the error in the phase angle is very small for the approximations C_2 and C_3, conditions for stability for any feedback system will be determined with quite high accuracy using these two approximations.

The error in predicting the response in the time domain when using the approximation in the frequency domain can be studied by a number of techniques. One of these is based upon Parseval's theorem which states

$$\int_0^\infty e^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} e(s)\,e(-s)\,ds = \frac{1}{\pi}\int_0^\infty |e(j\omega)|^2 d\omega \tag{29}$$

In other words, the integral of the square error in the time domain is equal to the integral of the square error in the frequency domain. The latter is easily derived for the above approximations ($h_a(j\omega)$) as illustrated in Fig. 11.

$$\begin{aligned} e(j\omega) &= (e^{-j\tau\omega} - h_a(j\omega))u(j\omega) \\ &= e^{-j\tau\omega}(1 - e^{j(\angle h_a + \tau\omega)})u(j\omega) \\ &= e^{-j\tau\omega}(1 - (\cos\Delta\varphi(\omega) + j\sin\Delta\varphi(\omega)))u(j\omega) \end{aligned} \tag{30}$$

Applying this to (29) we get

$$\begin{aligned} \int_0^\infty e^2(t)\,dt &= \frac{1}{\pi}\int_0^\infty 2(1 - \cos\Delta\varphi(\omega)|u(j\omega)|^2 d\omega \\ &= \frac{4}{\pi}\int_0^\infty \left(\sin\frac{1}{2}\Delta\varphi(\omega)\right)^2 |u(j\omega)|^2 d\omega \end{aligned} \tag{31}$$

The last term under the integral for a step function and a 'soft' step function respectively will be

$$|u(j\omega)|^2 = \frac{1}{\omega^2} \tag{32}$$

and

$$|u(j\omega)|^2 = \frac{1}{\omega^2(1+\tau^2\omega^2)} \tag{33}$$

Using (7) it follows that

$$\frac{1}{2}\Delta\varphi(\omega) = -\frac{\tau\omega}{2} + \arctan\ \omega c_1 \frac{1 - \dfrac{c_3}{c_1}\omega^2}{1 - c_2\omega^2} \tag{34}$$

for approximation C_3 and similarly for approximations A_3 and B_3. When $\omega \to 0$, we see that

$$\frac{1}{2}\Delta\varphi(\omega) \approx -\omega\left(\frac{\tau}{2} - c_1\right) \tag{35}$$

which when applied to (31) shows that the integrand will approach $(\tau/2 - c_1)^2$ when $\omega \to 0$. The similar expression for approximations A_3 and B_3 will be 0 since $a_1 = b_1 = \tau/2$. Thus the contribution to the integral of (31) will be slightly higher at the lowest frequencies for approximation C_3, but at higher frequencies the contribution to the integral will be smaller using approximation C_3 rather than A_3 or B_3.

5. Conclusions

A new rational transfer function approximation for transport delay has been developed and compared with the approximations most commonly used in the literature. It has been shown that the new approximation is superior particularly when applied to the determination of conditions for stability in feedback control systems containing elements with delay.

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