MODELING, IDENTIFICATION AND CONTROL, 1990, VOL. 11, NO. 2, 123–125
doi:10.4173/mic.1990.2.5

A note on a necessary condition for optimality

DAVID DI RUSCIO†

Keywords: *Linear optimal control, multivariable systems, pole placements, eigenvalues, control theory.*

A necessary condition is derived for the optimality of a linear multivariable feedback control system with respect to a quadratic performance index of infinite settling time.

1. Introduction

In linear quadratic system design the relation between the quadratic weights and the dynamic characteristics of the closed-loop system is of interest. This problem has been solved for a second order system (Di Ruscio and Balchen, 1990). In the general case, very little is known about these relations.

Let λ_i, $i=1,\dots,n$ be the open-loop poles and s_i, $i=1,\dots,n$ be the closed-loop poles. Mac Farlane (1970) presented an inequality between the product of the open-loop poles and the product of the closed-loop poles,

$$\prod_{i=1}^{n} |s_i| \geq \prod_{i=1}^{n} |\lambda_i|$$

Koussiouris (1982) (later corrected by Amin, 1984) presented an inequality between the sum of squares of the open-loop poles and closed-loop system poles,

$$\sum_{i=1}^{n} s_i^2 \geq \sum_{i=1}^{n} \lambda_i^2$$

These relations are, as we can see, conservative because the quadratic weights do not appear in the relations. In this paper an exact relation between the open-loop and closed-loop system poles is presented. We will show that the relation by Amin and Koussiouris is a special case of our relation.

2. Theory

Let **A** be the $n \times n$ real state system matrix and **B** the $n \times r$ real control input matrix. Let **Q** be the $n \times n$ real symmetric state weight matrix and **P** the $r \times r$ real symmetric positive definite control input weight matrix. Then we have the following theorem.

Theorem

If s_i, $i=1,\dots,n$ are the closed-loop system poles and λ_i, $i=1,\dots,n$ are the open-loop system poles, then s_i, $i=1,\dots,n$ are related to A or λ_i, Q and the positive semidefinite matrix $H=BP^{-1}B^{\mathrm{T}}$ by the following equality,

$$\sum_{i=1}^{n} s_i^2 = \mathrm{tr}(A^2) + \mathrm{tr}(HQ) \tag{1}$$

Received 1 August, 1990.

† Division of Engineering Cybernetics, Norwegian Institute of Technology, N-7034 Trondheim, Norway.

or equivalently

$$\sum_{i=1}^{n}\{(\mathrm{Re}\,s_i)^2-(\mathrm{Im}\,s_i)^2\}=\sum_{i=1}^{n}\{(\mathrm{Re}\,\lambda_i)^2-(\mathrm{Im}\,\lambda_i)^2\}+\mathrm{tr}(HQ) \tag{2}$$

Proof

The Hamiltonian matrix is derived from optimal control theory by augmenting the co-states to the state space model. The Hamiltonian is

$$F=\begin{bmatrix} A & -H \\ -Q & -A^{\mathrm{T}} \end{bmatrix}\in \mathbf{R}^{2nx2n} \tag{3}$$

F has $2n$ eigenvalues (n stable and n unstable) located symmetrically about the imaginary axis. The stable eigenvalues are identical to the eigenvalues of the closed-loop system matrix $(A+BG)=(A-HR)$, where G is the feedback matrix and R the symmetric solution of the algebraic Riccati equation. This can be seen from the following similarity transformation

$$\begin{bmatrix} I & 0 \\ R & I \end{bmatrix}^{-1}\begin{bmatrix} A & -H \\ -Q & -A^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I & 0 \\ R & I \end{bmatrix}=\begin{bmatrix} A-HR & -H \\ 0 & -(A-HR)^{\mathrm{T}} \end{bmatrix} \tag{4}$$

This means that the closed-loop eigenvalue spectrum can be derived from F without solving the algebraic Riccati equation.

Let $\rho(F)$ denote the spectrum (set of $2n$ eigenvalues) of F. Then we have that

$$\rho_i(F)=s_i,\ i=1,\dots,2n \tag{5}$$

$$s_i=-s_{n+i},\ i=1,\dots,n \tag{6}$$

$$\mathrm{tr}(F)=\sum_{i=1}^{2n}s_i=0 \tag{7}$$

We also have tha

$$\rho_i(F^2)=s_i^2,\ i=1,\dots,2n \tag{8}$$

$$s_i^2=s_{n+i}^2,\ i=1,\dots,n \tag{9}$$

$$\mathrm{tr}(F^2)=2\sum_{i=1}^{n}s_i^2 \tag{10}$$

We now need an expression for $\mathrm{tr}(F^2)$. We have

$$F^2=\begin{bmatrix} A^2+HQ & HA^{\mathrm{T}}-AH \\ A^{\mathrm{T}}Q-QA & (A^2+HQ)^{\mathrm{T}} \end{bmatrix} \tag{11}$$

From (11) we have

$$\mathrm{tr}(F^2)=\mathrm{tr}(A^2+HQ)+\mathrm{tr}((A^2+HQ)^{\mathrm{T}})=2\mathrm{tr}(A^2+HQ) \tag{12}$$

From (10) and (12) we get

$$\sum_{i=1}^{n}s_i^2=\mathrm{tr}(A^2)+\mathrm{tr}(HQ) \tag{13}$$

and the theorem is proved.

Q can be written as $Q=S^TS$ ($Q\geqslant 0$ and rank $(S)=$ rank (Q)). Then we have that

$$\mathrm{tr}(HQ)=\mathrm{tr}(HS^TS)=\mathrm{tr}(SHS^T)\geqslant 0 \tag{14}$$

because the matrix SHS^T is symmetric and positive semidefinite. Combining (2) and (14) we deduce the inequality presented in Amin (1984) and partly in Koussiouris (1982), i.e.

$$\sum_{i=1}^{n}\{(\mathrm{Re}\, s_i)^2-(\mathrm{Im}\, s_i)^2\}\geqslant\sum_{i=1}^{n}\{(\mathrm{Re}\,\lambda_i)^2-(\mathrm{Im}\,\lambda_i)^2\} \tag{15}$$

An alternative way to establish the theorem is as follows. We have

$$(A-HR)^2=(A-HR)(A-HR)=A^2+H(-RA+RHR)-AHR \tag{16}$$

Combining equation (16) and the algebraic Riccati equation gives

$$(A-HR)^2=A^2+HQ+(HA^T-AH)R \tag{17}$$

Take the trace on both sides of equation (17), note that $\mathrm{tr}[(HA^T-AH)R]=0$, and equation (1) is proved.

The same procedure as above can be used to determine expressions for

$$\sum_{i=1}^{n} s_i^{2k},\ k=1,2,3,\ldots$$

For $k=2$ we have

$$\begin{aligned}\sum_{i=1}^{n} s_i^4&=\sum_{i=1}^{n}\{((\mathrm{Re}\, s_i)^2-(\mathrm{Im}\, s_i)^2)^2-4(\mathrm{Re}\, s_i)^2(\mathrm{Im}\, s_i)^2\}\\&=\mathrm{tr}(A^4+4QA^2H-2QAHA^T+(HQ)^2)\\&=\mathrm{tr}[(A^2+HQ)(A^2+HQ)+(A^TQ-QA)(HA^T-AH)]\end{aligned} \tag{18}$$

Finally, note that this procedure cannot be used to determine expressions for

$$\sum_{i=1}^{n} s_i^{2k-1},\ k=1,2,3,\ldots \tag{19}$$

because

$$\mathrm{tr}(F^{2k-1})=0,\ k=1,2,3,\ldots \tag{20}$$

3. Conclusions

A necessary condition is presented for the optimality of a linear multivariable feedback control system with respect to a quadratic performance index of infinite settling time.

References

AMIN, M. A. (1984). Further comments on 'A necessary condition for optimization in the frequency domain' and on 'Optimization and pole placement for a single input controllable system'. *International Journal of Control*, **40**, 863–865.

DI RUSCIO, D., and BALCHEN, J. G. (1990). A Schur method for designing LQ-optimal systems with prescribed eigenvalues. *Modeling, Identification and Control*, **11**, 55–72.

KOUSSIOURIS, T. G. (1982). A necessary condition for optimization in the frequency domain. *International Journal of Control*, **36**, 213–215.

MACFARLANE, A. G. J. (1970). Two necessary conditions in the frequency domain for the optimality of a multiple-input linear control system. *Proceedings IEEE*, **117**, 464–466.