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The stability of 2 × 2 multivariable control systems

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The paper develops simple but powerful expressions for the investigation of stability and response of 2 × 2 multivariable control systems. The analytical results can be given convenient graphical representations. This provides a helpful tool for obtaining deep insight into the behaviour of this type of system which is particularly important in process control. Two simple examples illustrate the application of the method.

1. Introduction

When designing and tuning multivariable control systems simple rules of thumb are needed which can be given straightforward graphical interpretations. Since the 2 × 2 system (two controllers) is the most usual multivariable system, it is particularly important to be able to control this type of system. Most process control systems have a structure where the process transfer matrix is quadratic (same number of control variables as measurements) and the simplest case is when the transfer matrix of the controller is diagonal (multiple monovariable controllers). The problem of pairing variables in multivariable control has been dealt with by many authors and has reached a satisfactory level of resolution (Balchen 1963, Bristol 1966, McAvoy 1983, Balchen and Mumme 1988). Some of the pairing techniques only consider static, non-dynamic conditions and are not especially concerned with the stability of the total system. This is unsatisfactory. The present note considers the stability of a 2 × 2 multivariable system.

2. Stability of 2 × 2 systems

The solution given below was first presented in Balchen (1958) and has appeared in a number of text books (for example Balchen 1963) and papers (for example Rijnsdorp 1965) but does not appear to have been properly recognized.

The system shown in Fig. 1 is considered where

$H_u(s)$: process transfer matrix

$H_c(s)$: controller transfer matrix

The stability of this multivariable loop is determined by the location of the zeros of the complex function

$$\det(I + H_u(s)H_c(s)) \tag{1}$$

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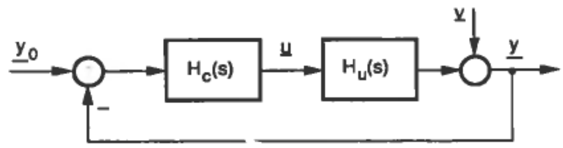

Figure 1. A general multivariable feedback control system.

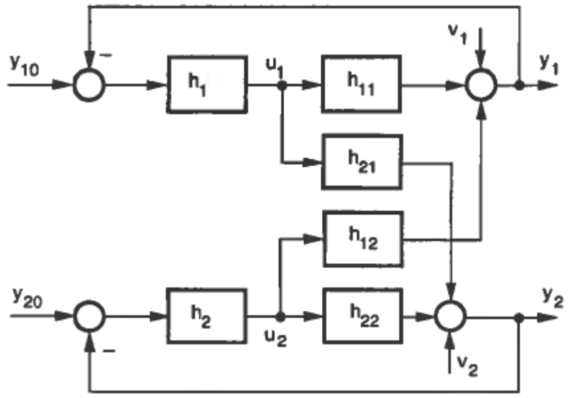

Figure 2. A 2×2 multivariable feedback control system with two monovariable controllers (diagonal controller).

For the 2×2 case we define as in Fig. 2

$$H_u(s)=\begin{bmatrix} h_{11}(s)h_{12}(s) \\ h_{21}(s)h_{22}(s) \end{bmatrix} \tag{2}$$

$$H_c(s)=\begin{bmatrix} h_1(s) & 0 \\ 0 & h_2(s) \end{bmatrix} \tag{3}$$

Introducing (2) and (3) into (1) we obtain

$$\det(I+H_u(s)H_c(s))=(1+h_1h_{11})(1+h_2h_{22})\left(1-\frac{h_{12}h_{21}}{h_{11}h_{22}}M_1M_2\right) \tag{4}$$

where

$$M_1=\frac{h_1h_{11}}{1+h_1h_{11}} \tag{5}$$

and

$$M_2=\frac{h_2h_{22}}{1+h_2h_{22}} \tag{6}$$

According to the Nyquist stability criterion the system is asymptotically stable if the expression of (4) does not have zeros in the right half of the complex plane. The first two brackets of (4) reflect the two individual monovariable control systems which would result if the cross coupling in the process did not exist ($h_{12}=0$ or $h_{21}=0$). In other words, these two individual systems have to be independently stable. The last term of (4) represents a requirement in which the cross-coupling in the process is involved. The expressions $M_1(s)$ and $M_2(s)$ represent the control tracking ratios of the two monovariable control systems in case of no cross-coupling. Furthermore the expression

$$Y(s)=\frac{h_{12}(s)h_{21}(s)}{h_{11}(s)h_{22}(s)} \tag{7}$$

appears. This transfer function expresses the degree of cross coupling in the process.

The third requirement for stability of the system is that the expression of the last term of (4) has no zeros in the right half of the complex plane. Equivalently it can be required that the expression

$$\left(-\frac{1}{Y(s)}+M_1(s)M_2(s)\right) \tag{8}$$

is without zeros in the right half of the complex plane.

This condition can be studied using the frequency response version of the Nyquist stability criterion by replacing $s=j\omega$. Then it will be a matter of the motion of the locus of

$$M_1(j\omega)M_2(j\omega)$$

relative to the locus of

$$\frac{1}{Y(j\omega)}.$$

Figure 3 shows a polar plot of these two complex functions and the vector corresponding to (8) with $s=j\omega$ is drawn. The Nyquist criterion expresses that the M_1M_2 locus is not to encircle the corresponding point (same frequency) of the $1/Y$ locus. The situation in Fig. 3 with a dotted curve section of $1/Y$ presents a stable system. The other case with fully drawn locus and intersection between the two loci illustrates a potentially unstable situation. But instability will only be detected if the point on the locus $1/Y(j\omega)$ is 'caught' inside the locus of $M_1(j\omega)M_2(j\omega)$. Conditions for this situation will be that $\omega'_M<\omega'_Y$ and $\omega''_Y<\omega''_M$ or $\omega'_Y<\omega'_M$ and $\omega''_M<\omega''_Y$ where ω'_M etc. are frequencies at the intersection points of the two loci as shown in Fig. 3.

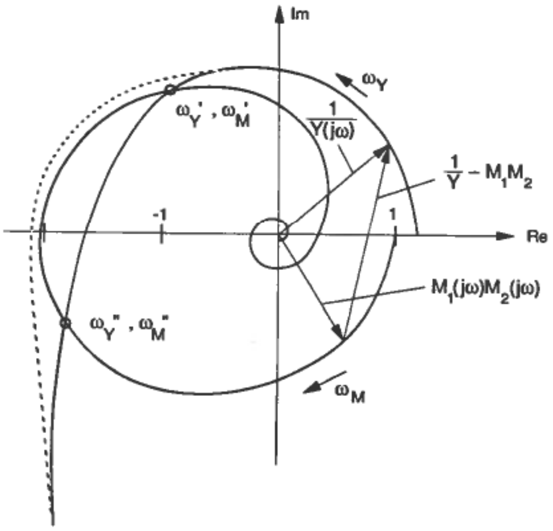

Figure 3. Polar plot of the relationship between $1/Y(j\omega)$ and $M_1(j\omega)M_2(j\omega)$.

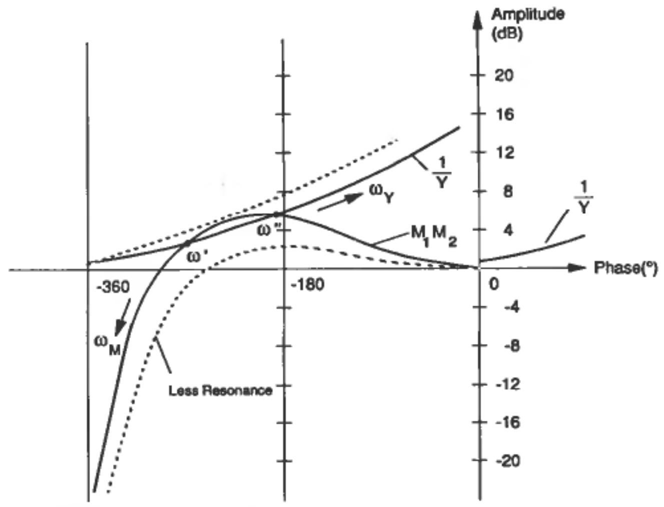

Figure 4. Cartesian phase-dB presentation of the relationship between $1/Y(j\omega)$ and $M_1(j\omega)M_2(j\omega)$.

These conditions apply to the particular configuration in Fig. 3. Other cases may occur when the two loci move relative to each other in a different way yielding other inequalities than those stated above.

It will usually be advantageous to do the graphical construction as in Fig. 3 in a 'log amplitude (dB) versus angle' Cartesian diagram. This is illustrated in Fig. 4. The same type of loci as in Fig. 3 are redrawn in Fig. 4 and the same types of intersections are seen to occur.

If an analysis of Fig. 4 indicates that the system is unstable, it is immediately recognized what must be done in order to achieve stability. Since the locus $1/Y(j\omega)$ is independent of the controller settings, it is clear that the only way to 'disentangle' the two loci is to reduce the resonance peak of $M_1(j\omega)$ or $M_2(j\omega)$, or both, by adjusting the proper controller parameters. This is indicated in Fig. 4 by an M_1M_2-locus with less resonance.

3. Examples

Example 1

A distillation column studied by Toijala and Fagervik (1972) as given in Fig. 5 will illustrate the method. The distillation column has 11 trays. The mass balance controls at the bottom and in the accumulator are assumed to be ideal and the concentration of the top and bottom products are to be controlled via measurements on trays 8 and 3 respectively and by manipulating the reflux rate (R) and the reboiler rate (V).

Based upon a definition of the control vector as

$$\boldsymbol{u}=\begin{bmatrix} R \\ V \end{bmatrix}$$

and the measurement vector

$$\boldsymbol{y}=\begin{bmatrix} x_8 \\ x_3 \end{bmatrix}$$

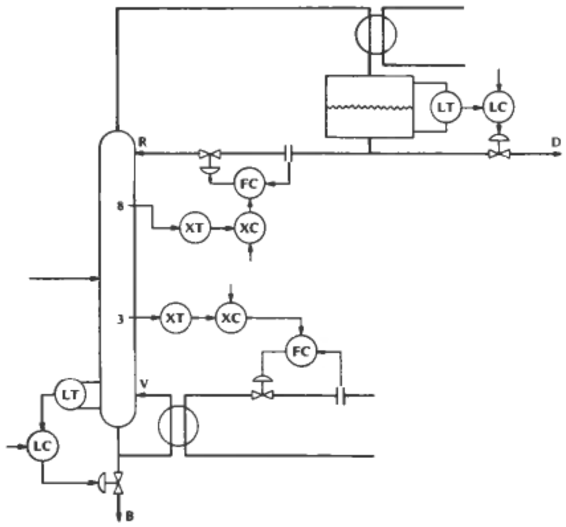

Figure 5. 2 × 2 control of product composition in a distillation column via reflux and boilup rate (Toijala and Fagervik 1972).

we get the elements of the transfer matrix

$$h_{11}(s)=\frac{0{\cdot}673}{(1+0{\cdot}05s)(1+12{\cdot}2s)(1+0{\cdot}083s)^3} \tag{9}$$

$$h_{12}(s)=\frac{-0{\cdot}575\exp(-0{\cdot}12s)}{(1+0{\cdot}05s)(1+0{\cdot}167s)(1+11{\cdot}5s)(1+0{\cdot}083s)} \tag{10}$$

$$h_{21}(s)=\frac{0{\cdot}462}{(1+0{\cdot}05s)(1+12{\cdot}2s)(1+0{\cdot}083s)^8} \tag{11}$$

$$h_{22}(s)=\frac{-0{\cdot}488\exp(-0{\cdot}03s)}{(1+0{\cdot}05s)(1+0{\cdot}167s)(1+11{\cdot}5s)(1+0{\cdot}083s)} \tag{12}$$

These transfer functions contain information about the control valves and the measuring units in addition to the dynamic behaviour of the distillation column.

Utilizing these transfer functions, we find

$$Y(s)=\frac{h_{12}(s)h_{21}(s)}{h_{11}(s)h_{22}(s)}=\frac{0{\cdot}8089\exp(-0{\cdot}09s)}{(1+0{\cdot}083s)^5} \tag{13}$$

The plots shown in Fig. 6 can be drawn based upon the analysis in Balchen and Mumme (1988). The tuning of the two controllers were such that the resonance peaks of $M_1(j\omega)$ and $M_2(j\omega)$ were about 3 dB. The result in Fig. 6 is that this system is just about unstable. In order to arrive at a system with an acceptable margin of stability, the two loops should be given a tuning with less resonance for example by reducing the controller proportional gain. Thereby the bandwidth of the control systems will also be reduced.

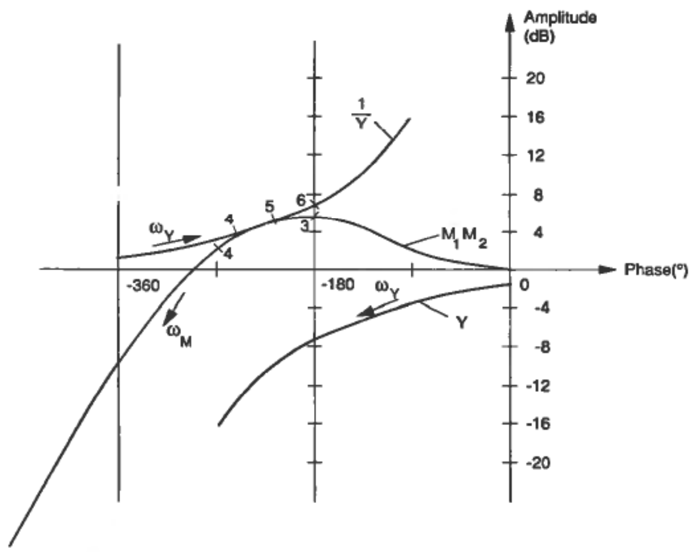

Figure 6. Phase-dB diagram for $1/Y - M_1M_2$ for distillation column with nominal controller tuning.

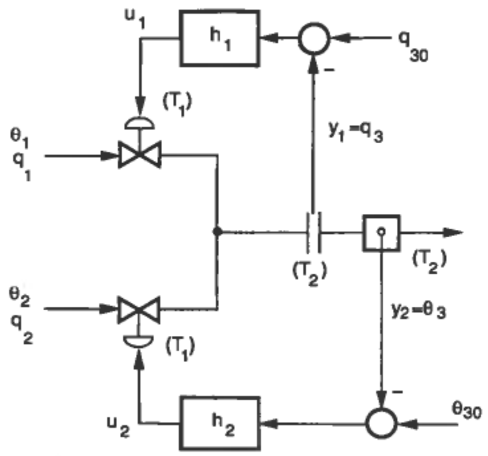

Figure 7. 2×2 control system for flow and temperature control of water mixing process.

This result is not at all obvious and could not be seen in any simple manner by observing the two control systems individually. For operational reasons it is a great advantage to be able to tune one control loop at a time while the other is disconnected (in manual mode). Further, the system is to maintain stability irrespective of whether one or two loops are in operation.

Example 2

A well-known example of a 2×2 multivariable system is that given in Fig. 7. It illustrates the control system for mixing two water flows, one hot and one cold. The objective is to maintain a constant output flow (q_3) and output temperature (θ_3). In setting up a simple mathematical model of this system, it is assumed that the two control valves are linear, have constant pressure drops and are driven by motors with identical time constants (T_1). Furthermore it is assumed that the measuring units for flow and temperature are linear and have the same time constant (T_2). Using the notation previously introduced we then get

$$h_{11}(s)=\frac{\partial q_3}{\partial q_1}\frac{1}{(1+T_1s)(1+T_2s)} \tag{14}$$

$$h_{22}(s)=\frac{\partial \theta_3}{\partial q_2}\frac{1}{(1+T_1s)(1+T_2s)} \tag{15}$$

$$h_{12}(s)=\frac{\partial q_3}{\partial q_2}\frac{1}{(1+T_1s)(1+T_2s)} \tag{16}$$

$$h_{21}(s)=\frac{\partial \theta_3}{\partial q_1}\frac{1}{(1+T_1s)(1+T_2s)} \tag{17}$$

where

$$\frac{\partial q_3}{\partial q_1}=\frac{\partial q_3}{\delta q_2}=1 \tag{18}$$

$$\frac{\partial \theta_3}{\partial q_1}=\frac{(\theta_1-\theta_2)q_2}{q_3^2}=\frac{\theta_1-\theta_{30}}{q_{30}} \tag{19}$$

$$\frac{\partial \theta_3}{\partial q_2}=-\frac{(\theta_1-\theta_2)q_1}{q_3^2}=-\frac{\theta_{30}-\theta_2}{q_{30}} \tag{20}$$

If we choose the desired values to be

$$q_{30}=20\,(\mathrm{l/min}), \quad \theta_{30}=50\,(^\circ\mathrm{C})$$

and the supply water temperatures

$$\theta_1=90\,(^\circ\mathrm{C}), \quad \theta_2=10\,(^\circ\mathrm{C})$$

we will get the nominal values

$$\bar{q}_1=10\,(\mathrm{l/min}), \quad \bar{q}_2=10\,(\mathrm{l/min})$$

Thus we get

$$\frac{\partial \theta_3}{\partial q_1}=\frac{80\cdot 10}{400}=2 \quad \text{and} \quad \frac{\partial \theta_3}{\partial q_2}=-\frac{80\cdot 10}{400}=-2$$

Applying these values to (14)–(17), we obtain

$$Y(s)=\frac{\theta_1-\theta_{30}}{\theta_2-\theta_{30}}=-1 \tag{21}$$

It is observed that only the transfer functions of (15) and (17) are dependent upon the operating conditions of the system (desired temperature and flow and the supply water temperatures) as a result of (19) and (20). This means that the gain of the temperature control loop will change with the operating conditions according to (20) if the controller is constant.

Assuming the numerical values given above and $T_1 > T_2$ and furthermore PI-controllers tuned to a relative damping of $\zeta = 0{\cdot}5$ we will get

$$M_1(s) = M_2(s) \approx \frac{1}{1 + T_2 s + T_2^2 s^2} \tag{22}$$

(8), (21) and (22) lead to the graphical result shown in Fig. 8. According to this figure, the system is definitely unstable. This instability is caused by the control loops being tuned to a relative damping which is too low. Stability could have been achieved if the controller parameters were tuned so that each of the loops had more than critical damping ($\zeta > 1$). Another cause of the instability is that the two loops have identical frequency response. If one loop had been faster than the other, the two resonance peaks would not add to each other in the critical region around $\angle M_1 M_2 = -180°$ as seen in Fig. 8. One of the loops (say M_2) could well be tuned to resonance, the peak then occurring at $\angle M_2 = -90°$. If the other loop had its resonance at another frequency (say higher), then stability could have been achieved as indicated by the dotted locus in Fig. 8.

The 'critical point' ($1/Y(s)$) is -1 in Fig. 8 according to (21), But in fact it is dependent upon the supply water temperatures and the desired temperature of the mixture. Assuming the operating conditions of the system to be as stated above except for the temperature of the hot water supply we get

$$\frac{1}{Y(s)} = -\frac{40}{\theta_1 - 50}$$

With $\theta_1 = 60$ (°C) we get $1/Y(s) = -4 = 12$ dB, bringing the critical point in Fig. 8 far enough above the M_1M_2-locus to assure stability.

The conclusion to be drawn from these two simple examples is that the conditions for stability of a 2×2 system may become rather involved and a graphical plot like that in Figs 6 and 8 is a great help in understanding the internal mechanisms.

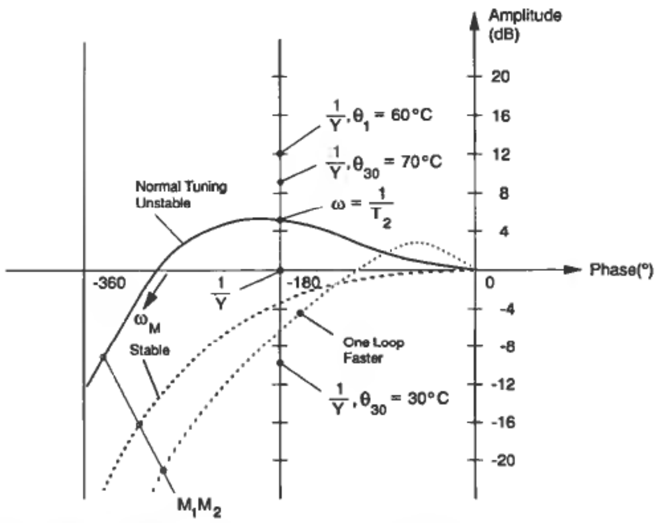

Figure 8. Phase-dB diagram for $1/Y - M_1M_2$ for water mixing process.

4. Response analysis of 2×2 systems

The response of the 2×2 multivariable system defined in Fig. 1 will now be dealt with relative to the response of an uncoupled system. Let us first consider the response of outputs $\boldsymbol{y}$ to a change in the setpoints $\boldsymbol{y}_0$. We introduce the notation for the coupled system.

$$\boldsymbol{y}=\bar{M}\boldsymbol{y}_0=\begin{bmatrix}\bar{M}_{11} & \bar{M}_{12}\\ \bar{M}_{21} & \bar{M}_{22}\end{bmatrix}\boldsymbol{y}_0 \tag{23}$$

After some algebraic manipulations it is found that

$$\bar{M}_{11}=M_1\frac{1-YM_2}{1-YM_1M_2} \tag{24}$$

$$\bar{M}_{22}=M_2\frac{1-YM_1}{1-YM_1M_2} \tag{25}$$

$$\bar{M}_{12}=\frac{h_{12}}{h_{22}}\frac{N_1M_2}{1-YM_1M_2} \tag{26}$$

$$\bar{M}_{21}=\frac{h_{21}}{h_{11}}\frac{N_2M_1}{1-YM_1M_2} \tag{27}$$

where

$$N_i=\frac{1}{1+h_ih_{ii}}$$

Referring to (24) we see that there is a correction in the uncoupled transfer function (M_1). The numerator term $(1-YM_2)$ represents the influence of the cross-coupling upon the appropriate process transfer function (h_{11}), whereas the denominator term $(1-YM_1M_2)$ gives the conditions for stability. The latter term is common to all transfer functions in (24)–(27). In (26) the term N_1 (the control deviation ratio) will be a small quantity and $|N_1(j\omega)|\to 0$ when $\omega\to 0$. Thus $|\bar{M}_{12}|\to 0$ when $\omega\to 0$. The same applies to (27). Thus the matrix M in (23) will tend to become a diagonal matrix at low frequencies as a result of the control structure.

The frequency response of the correction term of (24)

$$K_1(j\omega)=\frac{1-Y(j\omega)M_2(j\omega)}{1-Y(j\omega)M_1(j\omega)M_2(j\omega)} \tag{28}$$

will now be investigated.

It is observed that when the two control loops are active we will get $M_1(j\omega)\approx M_2(j\omega)\approx 1$, so that $K_1(j\omega)\approx 1$ and thus $\bar{M}_{11}(j\omega)\approx M_1(j\omega)$. Furthermore, when $|M_2(j\omega)|\ll 1$ we will have $K_1(j\omega)\approx 1$. Only in a particular frequency range will we have $K_1(j\omega)\neq 1$, that is when $|YM_2(j\omega)|$ and $|YM_1M_2(j\omega)|$ are different and around 1 or larger. In the system dealt with in Example 1 above, the factor $K_1(j\omega)$ will have a frequency response as derived from (28), as presented in Fig. 9.

Two cases are illustrated: (A) with normal individual loop tuning, ($|M|_{max}=3$ dB) and (B) with each controller gain reduced by 3 dB. It is observed that case (A) is very close to instability and a high resonance peak in the frequency response and that even case (B) has pronounced resonance.

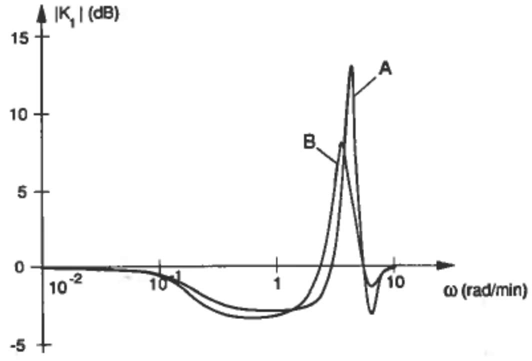

Figure 9. Frequency response of dynamic correction term $|K_1(j\omega)|$ of (28) for 2×2 distillation column control with different controller tunings. A: Each monovariable loop tuned to 3 dB resonance peak, B: Each controller gain reduced by 3 dB relative to case A.

The response of the 2×2 multivariable system to external disturbances characterized by

$$\boldsymbol{v}=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

will be denoted by

$$\boldsymbol{y}=\bar{N}\boldsymbol{v}=\begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix}\boldsymbol{v} \tag{29}$$

Some algebraic manipulations lead to

$$\bar{N}_{11}=N_1\frac{1}{1-YM_1M_2} \tag{30}$$

$$\bar{N}_{22}=N_2\frac{1}{1-YM_1M_2} \tag{31}$$

$$\bar{N}_{12}=-\frac{h_{12}}{h_{22}}\frac{N_1M_2}{1-YM_1M_2} \tag{32}$$

$$\bar{N}_{21}=-\frac{h_{21}}{h_{11}}\frac{N_2M_1}{1-YM_1M_2} \tag{33}$$

These results have very much in common with those in (24)–(27). In fact the results of (32) and (33) are identical to those of (26) and (27) as should have been expected. The results of (30) and (31) indicate that the disturbance reduction ratios $\bar{N}_{11}$ and $\bar{N}_{22}$ for the system with cross coupling is given by the disturbance reduction ratio for the uncoupled system multiplied by the correction factor

$$K_2=\frac{1}{1-YM_1M_2} \tag{34}$$

The structure of the disturbances assumed in (29) is special in that there is one independent disturbance acting upon each of the output variables. A more general case will be given by

$$\boldsymbol{y}(s)=\bar{N}(s)H_v(s)\boldsymbol{v}(s) \tag{35}$$

in which $H_v(s)$ is a process transfer matrix of the same form as (2). In order to be able to compare the disturbance response of a process with cross coupling with the response of one without, we introduce the 'uncoupled disturbance response'

$$\mathbf{y}_{nc}(s) = N(s)H_v(s)\mathbf{v}(s) \tag{36}$$

in which

$$N(s) = \begin{bmatrix} N_1(s) & 0 \\ 0 & N_2(s) \end{bmatrix} \tag{37}$$

Now we find

$$\mathbf{y}(s) = \bar{N}(s)N^{-1}(s)\mathbf{y}_{nc}(s) = K(s)\mathbf{y}_{nc}(s) \tag{38}$$

$K(s)$ is a matrix of correction factors

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where

$$K_{11} = K_{22} = \frac{1}{1 - YM_1M_2} = K_2 \tag{39}$$

$$K_{12} = -\frac{h_{12}}{h_{22}} \frac{N_1}{N_2} M_2 K_2 \tag{40}$$

$$K_{21} = -\frac{h_{21}}{h_{11}} \frac{N_2}{N_1} M_1 K_2 \tag{41}$$

(39) is particularly interesting because it determines the ratio between the direct disturbance response of the coupled and uncoupled systems.

5. Pairing of 2×2 systems

The question of pairing in multivariable systems is not dealt with explicitly in this note. Nevertheless, the pairing of 2×2 system is easily resolved by the method discussed above because there are only two choices. The result of this is that factor $Y(s)$ will either be as in (7) or its inverse. It is easy to make a choice between these two as illustrated in Fig. 6 where the frequency response of both functions are drawn. In that example it is immediately observed that the locus of $Y(j\omega)$ is always below the locus of $M_1(j\omega)M_2(j\omega)$, indicating structural instability. Thus the correct choice of pairs is made in Example 1.

In the system in Example 2 it is seen from Fig. 8 that the choice of pairs is indifferent because factor $Y(s)$ is independent of the pairing with the operating conditions given in the example. If the desired temperature of the mixture was set to $\theta_{30} = 70\,(^\circ\mathrm{C})$ the critical points would move upwards to $1/Y(s) = -3 = 9{\cdot}5$ dB indicating that the chosen pairs were correct. On the other hand, if $\theta_{30} = 30\,(^\circ\mathrm{C})$ the critical point would move to $1/Y(s) = -\frac{1}{3} = -9{\cdot}5$ dB indicating that the pairs should be interchanged in order to achieve a stable system.

6. Conclusions

The 2×2 systems are probably the most important multivariable systems in real life because systems of higher complexity may often be approximated by 2×2 systems. The method discussed in this note yields insight into the mechanisms of stability and the response of such systems in a way that is directly applicable to controller synthesis. The method is based upon a result which has been available for more than 30 years.

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