A Schur method for designing *LQ*-optimal systems with prescribed eigenvalues

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In this paper a new algorithm for solving the *LQ*-optimal pole placement problem is presented. The method studied is a variant of the classical eigenvector approach and instead uses a set of Schur vectors, thereby gaining substantial numerical advantages. An important task in this method is the *LQ*-optimal pole placement problem for a second order (sub) system. The paper presents a detailed analytical solution to this problem. This part is not only important for solving the general *n*-dimensional problem but also provides an understanding of the behaviour of an optimal system. The paper shows that in some cases it is an infinite number; in others a finite number; and in still others, non state weighting matrices *Q* that give the system a set of prescribed eigenvalues. Equations are presented that uniquely determine these state weight matrices as a function of the new prescribed eigenvalues. From this result we have been able to derive the maximum possible imaginary part of the eigenvalues in an *LQ*-optimal system, irrespective of how the state weight matrix is chosen.

1. Introduction

In this paper a new method for the determination of state weighting matrices, *Q*, is presented such that the closed loop system gets a set of prescribed eigenvalues and the feedback is a solution from the matrix Riccati equation. The problem of designing feedback gains to optimally place the poles can be divided into two classes.

The first class is to design the *LQ*-regulator by which all poles of the closed loop system are located in a specified region. This problem is discussed, among others, by Anderson and Moore (1969), Kawasaki and Shimemura (1983, 1988), Heger and Frank (1984), Shieh *et al.* (1986), Shieh *et al.* (1988).

Anderson and Moore (1969) placed the poles of the *LQ*-optimal closed loop system to the left of a vertical line on the negative real axis in the complex plane. Shieh *et al.* (1986) placed the eigenvalues in a vertical strip on the negative real axis. Kawasaki and Shimemura (1983, 1988) developed a recursive method for optimally placing the eigenvalues of the closed loop system in an open hyperbola of the complex plane, while Shieh *et al.* (1988) optimally placed the poles in an open sector of the complex plane. The hyperbola by Kawasaki *et al.* is an approximation of the sector by Shieh *et al.*

The second class (to be studied in this paper) is to design the *LQ*-regulator by which the closed loop system gets prescribed eigenvalues. This problem is discussed, among others, by Graupe (1972), Solheim (1972), Bar-Ness (1978), Juang and Lee (1984), Amin (1985), Medanic *et al.* (1988), Saif (1989), and Tharp (1989).

Solheim (1972) presented a method based on an eigenvector approach. This method performs well for a system with real distinct eigenvalues if the system is well behaved and therefore can be diagonalized from a numerical point of view. If the system has complex eigenvalues which have to be moved by Solheim's method, this is of little interest since the method neglects important solutions. If the system cannot be diagonalized (a necessary condition is that the system has multiple eigenvalues) the old algorithm cannot be used directly. The Jordan canonical form was proposed as a solution to this problem. But this is only of theoretical interest because the Jordan canonical form is very difficult to determine numerically, see for example Gulub (1983) pp. 196–198. It also turns out that the algorithm solves an unnecessary problem, namely the solution of full-order non-linear Riccati equations for modifying one real or two complex eigenvalues in each step.

Graupe (1972) and Bar-Ness (1978) presented methods based on the sensitivity of the state weight matrix with respect to the eigenvalues. Juang and Lee (1984) used the modal control approach by Varga (1981) to determine a feedback gain matrix such that the closed loop system gets specified eigenvalues. Then an inverse control approach (Kalman 1964) is used to determine a possible corresponding state weight matrix. The method is based on Theorem 1 and Corollary 1 of Juang and Lee (1984). Amin and Hassan (1985) shows that neither the theorem nor the corollary can be correct.

Amin (1985) considered the same problem as Solheim (1972) and developed a method based on the results of Anderson and Moore (1969), and the mirror-image property by Moliniari (1977). In this method it is only possible to change the real parts of the eigenvalues while the imaginary parts are preserved. This could lead to a requirement of the solution of a linear Lyapunov equation rather than a non-linear Riccati equation. Medanic *et al.* (1988) used the negative definite solutions of the Riccati equation to modify the state weight matrix and thereby position the real parts of the eigenvalues while the imaginary parts are preserved. Tharp (1989) used the method by Medanic *et al.* (1988) on linear discrete systems. Saif (1989) used the result by Solheim (1972), Medanic *et al.* (1988) and Amin (1985). In this method it is possible to change the real and the imaginary parts of the eigenvalues, however, there are unnecessary restrictions on the position of the new imaginary part of the eigenvalues. Saif used the results by Solheim to move complex eigenvalues. The method is therefore conservative, unnecessary restrictions are therefore put on the performance matrices. Nothing new is added in the paper by Saif.

The method presented here is an extension of the work by Solheim (1972). The method by Solheim (1972), and the other methods mentioned above, are not general with respect to the position of the new eigenvalues. One reason is that unnecessary restrictions are taken on the matrices in the performance criteria. The relation between these matrices and the closed loop eigenvalues are of interest. In this paper we will extend the relations given by Solheim, and present the general solution to this problem. From this result we have been able to derive the maximum possible imaginary part of the eigenvalues in an LQ-optimal system, irrespective of how the state weight matrix Q is chosen. This result completes the optimal pole placement problem, because it gives us the unique region where the poles can be located. In our method we are able to move two eigenvalues, real or complex conjugate, to new real or complex conjugate locations at each step, and only a Riccati equation of an order of not more than two has to be solved. The method may also be used to shift poles in an already optimal system to more desirable locations.

The Schur method presented here has considerable numerical advantages compared to the diagonalization approach. The algorithm is firstly, completely general to the system structure and secondly, completely general to the structure of the eigenvalue spectrum. The reduction to real Schur form is an intermediate step in computing eigenvectors (using the double Francis QR algorithm) so the Schur approach must, by definition, be faster. Furthermore, for a second order system the algorithm is fundamental, that means that all solutions (state weight matrices) Q are derived such that the closed loop spectrum gets specified eigenvalues.

The real Schur decomposition in control system design is used by Laub (1979) and Varga (1981). Laub (1979) used the real Schur decomposition to solve the algebraic Riccati equation, while Varga (1981) used the real Schur decomposition for solving the modal control problem, for example, designing a feedback gain without constraints such that the closed loop system gets prescribed eigenvalues.

The rest of the paper is organized as follows. Section 2 provides some definitions and theorems from linear algebra prior to obtaining the method. Section 3 provides some basics from control theory and the problem definitions are stated. Section 4 shows that the problem can be solved using a block triangulation approach. We show that the n-dimensional problem can be solved by decoupling a sub-system. Section 5 solves the problem for a second order general (sub) system. This is part of the n-dimensional problem. Equations are derived that determine all state weighting matrices Q (if there are any) that give the LQ-optimal system some prescribed eigenvalues. The results are presented in Algorithm 5.1. Section 6 states a general algorithm, Algorithm 6.1, for the n-dimensional problem. Numerous examples are given in Section 7 and some concluding remarks are made in Section 8.

2. Linear algebra review

Before presenting the result some definitions and theorems are prepared for obtaining the method. Proofs are given in Golub (1983) Ch. 7.

Definition 1. $U \in \mathbf{R}^{n \times n}$ is orthogonal if $U^T = U^{-1}$.

Definition 2. $U \in \mathbf{C}^{n \times n}$ is unitary if $U^H = U^{-1}$.

Definition 3. $F \in \mathbf{R}^{2n \times 2n}$ is Hamiltonian if $L^{-1}F^T L = -F$, where L is given below

$$L = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbf{R}^{2n \times 2n} \tag{1}$$

I denotes the $n \times n$ dimensional identity matrix. Note that $L^T = L^{-1} = -L$.

Theorem 1. Schur Decomposition (SD). Let $A \in \mathbf{R}^{n \times n}$. Then there exists a unitary similarity transformation U such that $U^H A U = T$ is upper triangular with the eigenvalues on the diagonal of T. Furthermore, U can be chosen such that the eigenvalues λ_i appear in any desired order along the diagonal.

Theorem 2. Real Schur Decomposition (RSD). Let $A \in \mathbf{R}^{n \times n}$. Then there exists an orthogonal similarity transformation U such that $U^T A U = T$ is quasi upper triangular. The diagonal blocks in T consist of 1×1 blocks corresponding to real eigenvalues and 2×2 blocks corresponding to complex conjugate eigenvalues. Furthermore, U can be chosen such that the 2×2 and 1×1 diagonal blocks appear in any desired order.

3. Basic definition

Consider a time invariant dynamic system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{2}$$

and the quadratic objective functional

$$J = \frac{1}{2} \int_0^\infty (\underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u}) \, \mathrm{d}t \tag{3}$$

where $\underline{x}$ is an n-dimensional state vector, $\underline{u}$ is an r-dimensional control input vector, A and B are constant matrices of appropriate dimensions and Q and P are $n \times n$ symmetrical and $r \times r$ positive definite symmetrical matrices respectively. The problem is to design a controller matrix G of dimension $r \times n$ so that the closed loop system (4) gets a set of prescribed eigenvalues and such that the performance criterion (3) is minimized

$$\dot{\underline{x}} = (A + BG)\underline{x} \tag{4}$$

The optimal control that minimizes criterion (3) is given by

$$\underline{u} = G\underline{x} \qquad G = -P^{-1}B^T R \tag{5}$$

where R is a solution to the algebraic Riccati equation

$$A^T R + RA - RHR + Q = 0, \qquad H = BP^{-1}B^T \tag{6}$$

The Hamiltonian matrix is derived from optimal control theory by augmenting the co-states to the state space model (2). The Hamiltonian is

$$F = \begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix} \in \mathbf{R}^{2n \times 2n} \tag{7}$$

F has $2n$ eigenvalues, (n stable and n unstable) located symmetrically about the imaginary axis. The stable eigenvalues are identical to the eigenvalues of the closed loop system matrix $(A + BG) = (A - HR)$. This can be seen from the following similarity transformation

$$\begin{bmatrix} I & 0 \\ R & I \end{bmatrix}^{-1} \begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ R & I \end{bmatrix} = \begin{bmatrix} A - HR & -H \\ 0 & -(A - HR)^T \end{bmatrix} \tag{8}$$

This means that the closed loop eigenvalue spectrum can be derived from F without solving the algebraic Riccati equation. We can now transform the eigenvalue problem by similarity transformation. Using the following orthogonal similarity transformation we get

$$\tilde{F} = \bar{U} F \bar{U}^T = \begin{bmatrix} U^T & 0 \\ 0 & U^T \end{bmatrix} \begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} T & -\tilde{H} \\ -\tilde{Q} & -T^T \end{bmatrix} \tag{9}$$

where T, $\tilde{Q}$ and $\tilde{H}$ are given by

$$T = U^T A U \qquad \tilde{Q} = U^T Q U \qquad \tilde{H} = U^T H U \tag{10}$$

This means that the LQ-problem in (2) and (3) is transformed to a new state space by the transformation $\underline{x} = U\underline{z}$ where U is an orthogonal matrix and $\underline{z}$ is the new state vector. The new system matrix T which is similar to A with respect to the eigenvalues, the new state weight matrix $\tilde{Q}$ is given in (10). The control input matrix for this system is $U^T B$.

Choosing U as an orthogonal matrix that transforms A by similarity transformation to an upper triangular Schur form T, then the eigenvalue dependence upon the state weighting matrix Q can be derived from $\tilde{F}$ in (9). This will be shown in the next section.

4. Solution by block triangulization

The eigenvalue dependence as a function of Q in an LQ-optimal system can be derived from the following Hamiltonian matrix (canonical system)

$$\tilde{F} = \begin{bmatrix} T & -\tilde{H} \\ -\tilde{Q} & -T^T \end{bmatrix} \tag{11}$$

the system matrix T, now in real Schur form (RSD) and $\tilde{H}$ can be written

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \in \mathbf{R}^{n \times n} \qquad \tilde{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \in \mathbf{R}^{n \times n} \tag{12}$$

T_{11} is an $n\text{-}2 \times n\text{-}2$ dimensional RSD matrix with 1×1 or 2×2 diagonal blocks. The 1×1 blocks correspond to real eigenvalues and the 2×2 blocks to complex conjugate eigenvalues. The matrix T_{22} is a 2×2 dimensional RSD block of either two 1×1 diagonal blocks or a general real 2×2 block. We note that if T_{22} consists of 1×1 blocks then the system modus number n in T is decoupled. Then λ_n can be moved directly by Solheim's algorithm (Solheim 1972). This is only a special case of our algorithm.

Block T_{22} is decoupled from the other elements in T. The eigenvalues of the general 2×2 block T_{22} can be moved while the other eigenvalues are unchanged by the following state weight matrix Q. Proof is given below.

$$\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix} \tag{13}$$

where Q_{22} is a 2×2 symmetric matrix (not necessary non-negative). We will now derive expressions for the eigenvalues of (11). The eigenvalues of $\tilde{F}$ are given by the characteristic polynomial of (11)

$$\det(sI - \tilde{F}) = \det \begin{bmatrix} sI - T_{11} & -T_{12} & H_{11} & H_{12} \\ 0 & sI - T_{22} & H_{12}^T & H_{22} \\ 0 & 0 & sI + T_{11}^T & 0 \\ 0 & Q_{22} & T_{12}^T & sI + T_{22}^T \end{bmatrix} \tag{14}$$

If we develop the determinant (14) from the first column and the third row we get the expression

$$\det(sI - \tilde{F}) = \det(sI - T_{11})\det(sI + T_{11}^T)\det(sI - F_{22}) \tag{15}$$

where F_{22} is a 4×4 Hamiltonian matrix corresponding to the decoupled sub-system T_{22} given by

$$F_{22} = \begin{bmatrix} T_{22} & -H_{22} \\ -Q_{22} & -T_{22}^T \end{bmatrix} \tag{16}$$

Using the Schur formula (given in the Appendix) on the third part on the right-hand side in equation (15) we get an alternative expression

$$\det(sI - \tilde{F}) = \det[(sI - T_{11})(sI + T_{11}^T)(sI - T_{22})(sI + T_{22}^T - Q_{22}(sI - T_{22})^{-1}H_{22})] \quad (17)$$

Before we continue we will mention a special case. Assume the scalar case where $T_{22} = \lambda_n$, $Q_{22} = q_{nn}$ and $H_{22} = h_{nn}$. Then from (15) or (17) we get

$$q_{nn} = \frac{s_n^2 - \lambda_n^2}{h_{nn}}, \quad h_{nn} \neq 0 \quad (18)$$

This was the main result in Solheim's (1972) algorithm.

The problem is now reduced to determining the eigenvalue dependence upon the weight matrix Q_{22} in the general 4×4 dimensional Hamiltonian matrix F_{22}. In the next section we will derive analytic expressions for the state weight matrix Q_{22} such that the decoupled sub-system in position T_{22} gets prescribed eigenvalues. These results are very important for studying the behaviour of an optimal system and do not seem to be known, or mentioned in the literature. Another important conclusion from this result is that when $\tilde{Q}$ has the form (13) and T and $\tilde{H}$ have the form in (12) then we do not have to solve a Riccati equation whose order is more than the dimension of the decoupled system (16). We also recall that there exists an orthogonal U such that the diagonal blocks in T appear in any desired order. This means that any diagonal block T_{ii} in T_{11} whose eigenvalues are to be moved can be placed in position T_{22} with an orthogonal similarity transformation (Stewart 1976).

5. The second order problem

In this section we will derive analytic expressions for the problem of determining state weight matrices Q such that a second order optimal system gets two new prescribed eigenvalues.

The Hamilton matrix that determines the closed loop eigenvalues is given by

$$F = \begin{bmatrix} T & -H \\ -Q & -T^T \end{bmatrix} \quad (19)$$

where T, Q and H are given by

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \qquad H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \qquad Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \quad (20)$$

The characteristic polynomial of the Hamiltonian matrix F is given by

$$L(s) = s^4 - p_1 s^2 + p_2 = 0 \quad (21)$$

The polynomial coefficients in terms of the state weight matrix elements are given by

$$p_1 = b_1 q_{11} + b_2 q_{22} + d_1 q_{12} + d_2 \quad (22)$$

$$p_2 = a_0 q_{11} q_{22} + a_1 q_{11} + a_2 q_{22} - a_0 q_{12}^2 + k_1 q_{12} + k_2 \quad (23)$$

where the coefficients are only dependent on the matrices T and H. The coefficients are given in the Appendix. The eigenvalues can either consist of real or complex conjugate

pairs. If we prescribe the eigenvalues of the Hamiltonian, the polynomial coefficients in (21) for these two cases would be

Case 1. Real eigenvalues λ_1 and λ_2

$$p_1 = \lambda_1^2 + \lambda_2^2 \tag{24}$$

$$p_2 = \lambda_1^2 \lambda_2^2. \tag{25}$$

Case 2. Complex conjugate eigenvalues $\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$

$$p_1 = 2(\alpha^2 - \beta^2) \tag{26}$$

$$p_2 = (\alpha^2 + \beta^2)^2 \tag{27}$$

We now have the problem of solving two equations with three unknowns. We choose q_{12} as a free parameter and get the following equations from (22) and (23) that determine q_{11} and q_{22}

$$a_0 q_{11} q_{22} + a_1 q_{11} + a_2 q_{22} + a_3 = 0 \tag{28}$$

$$b_1 q_{11} + b_2 q_{22} + b_3 = 0 \tag{29}$$

where a_3 and b_3 are functions of q_{12} and are given by

$$a_3 = -a_0 q_{12}^2 + k_1 q_{12} + k_2 - p_2 \tag{30}$$

$$b_3 = d_1 q_{12} + d_2 - p_1 \tag{31}$$

We will first discuss the case when $a_0 = \det H \neq 0$. Given q_{12} as a free parameter we can solve equations (28) and (29) for q_{11} and q_{22}. These equations could be written

$$a_0 b_1 q_{11}^2 + (a_0 b_3 - a_1 b_2 + a_2 b_1)q_{11} + a_2 b_3 - a_3 b_2 = 0 \tag{32}$$

$$a_0 b_2 q_{22}^2 + (a_0 b_3 + a_1 b_2 - a_2 b_1)q_{22} + a_1 b_3 - a_3 b_1 = 0 \tag{33}$$

We are only interested in the real solutions. These equations have real solutions if and only if (surprisingly we get only one inequality from equations (32) and (33))

$$\begin{aligned} J_l(q_{12}) &= (a_0 b_3 - a_1 b_2 + a_2 b_1)^2 - 4 a_0 b_1 (a_2 b_3 - a_3 b_2) \\ &= (a_0 b_3 + a_1 b_2 - a_2 b_1)^2 - 4 a_0 b_2 (a_1 b_3 - a_3 b_1) \\ &\geqslant 0 \end{aligned} \tag{34}$$

We recall that the coefficients a_3 and b_3 are functions of q_{12} while the others are constant given by t_{ij} and h_{ij}. If we develop (34) in terms of q_{12} we determine the fundamental inequality that constrains q_{12}. The inequality is given below.

$$J_l(q_{12}) = c_1 q_{12}^2 + c_2 q_{12} + c_3 \geqslant 0 \tag{35}$$

where the coefficient c_1 is constant and only dependent on the matrix elements in H. c_1 is easily determined by the equality below

$$c_1 = -4(\det H)^3 \tag{36}$$

When $\det H > 0$ then $J_l(q_{12})$ has a maximum value. This means that there must be a finite solution interval of (35) if there is any at all. If $\det H < 0$ then $J_l(q_{12})$ have a minimum value and there would be an infinite interval if there is any at all. But when $H = H^T < 0$ then the solution R of the Riccati equation is not necessarily positive

definite. The coefficients c_2 and c_3 are dependent on the coefficients of the new prescribed polynomial p_1 and p_2. Where the relation between the new prescribed eigenvalues and p_1 and p_2 is given in Case 1 and Case 2 above. We can now write coefficients c_2 and c_3 in terms of these coefficients

$$c_2 = g_1 p_1 + g_2 \tag{37}$$

$$c_3 = f_1 p_1^2 + f_2 p_1 + f_3 + f_4 p_2 \tag{38}$$

The coefficients g_i and f_i are again only dependent on T and H. See the Appendix for details.

If $a_0 = \det H = 0$, then the second order system of polynomial equations (28) and (29) will degenerate to a second order system of linear equations. The solution is given by

$$\begin{bmatrix} q_{11} \\ q_{22} \end{bmatrix} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 & -a_2 \\ -b_2 & a_1 \end{bmatrix} \begin{bmatrix} -k_1 q_{12} - k_2 + p_2 \\ -d_1 q_{12} - d_2 + p_1 \end{bmatrix} \tag{39}$$

Equation (39) can be written

$$q_{11} = v_1 q_{12} + v_2 \tag{40}$$

$$q_{22} = w_1 q_{12} + w_2 \tag{41}$$

where the coefficients v_i and w_i are given in the Appendix. When $a_1 b_2 - a_2 b_1 = 0$ then there is no solution to equation (39). Physically this means that the linear system model is uncontrollable (see comments in Section 6). We see from equations (40) and (41) that there will be an infinite number of solutions. However, if we require that $Q = Q^T \geqslant 0$ then we will get a finite number of solutions or none. In this case we will get an inequality $(q_{11} q_{22} - q_{12}^2 = c_4 q_{12}^2 + c_5 q_{12} + c_6 \geqslant 0)$ for q_{12} with the same form as (35). It is indeed clear that it could be necessary with a negative Q, for example when the new eigenvalues are specified right to the eigenvalues of the original system in the complex plane. Note that the eigenvalues can be specified all over the left complex plane, including the imaginary axis.

We can now summarize the above results in a general algorithm for a second order linear time invariant system. This algorithm is again part of the overall n-dimensional system in Section 6.

Algorithm 5.1

Step 1.   Given a second order system described by T and H. T can be a single system matrix and $H = BP^{-1}B^T$, or T could be a decoupled 2×2 block from a n-dimensional system and H the corresponding 2×2 diagonal block.

Step 2.   Compute coefficients a_i, b_i, c_1, d_i, g_i and f_i as functions of the elements of the matrices T and H. The coefficients are given in the Appendix.

Step 3.   Specify two eigenvalues. Compute the coefficients p_1 and p_2.

Step 4.   If $a_0 = \det H = 0$, then compute solutions from equations (40) and (41) and terminate, else jump to Step 5.

Step 5.   Compute coefficients c_2 and c_3 that depend on the new prescribed eigenvalues from

$$c_2 = g_1 p_1 + g_2$$

$$c_3 = f_1 p_1^2 + f_2 p_1 + f_3 + f_4 p_2$$

Step 6. Check if the proposed q_{12} satisfies the inequality

$$J_I(q_{12}) = c_1 q_{12}^2 + c_2 q_{12} + c_3 \geqslant 0$$

If this inequality is not satisfied, determine a q_{12} that does. If it is not possible to satisfy this inequality it means that it is not possible to determine LQ-optimal system with these pole locations. Recall that the real parts of the eigenvalues can be located all along the real axis, but that the imaginary value is physically limited. See equation (49) that uniquely determines the maximum possible imaginary value, no matter how Q are chosen.

Step 7. Compute coefficient b_3 as a function of q_{12}

$$b_3 = d_1 q_{12} + d_2 - p_1$$

Step 8. Compute the diagonal elements in Q. For each choice of q_{12} we get two generally different sets of weighting matrices. The two sets are marked with super index α and β. It is assumed that $a_0 \neq 0$ and that the system is controllable, then $b_1 \neq 0$ and $b_2 \neq 0$

$$\left. \begin{aligned} q_{11}^\alpha &= \frac{-a_0 b_3 + (a_1 b_2 - a_2 b_1) - \sqrt{(J_I(q_{12}))}}{2 a_0 b_1} \\[2mm] q_{22}^\alpha &= \frac{-a_0 b_3 - (a_1 b_2 - a_2 b_1) + \sqrt{(J_I(q_{12}))}}{2 a_0 b_2} \end{aligned} \right\} \tag{42}$$

$$\left. \begin{aligned} q_{11}^\beta &= \frac{-a_0 b_3 + (a_1 b_2 - a_2 b_1) + \sqrt{(J_I(q_{12}))}}{2 a_0 b_1} \\[2mm] q_{22}^\beta &= \frac{-a_0 b_3 - (a_1 b_2 - a_2 b_1) - \sqrt{(J_I(q_{12}))}}{2 a_0 b_2} \end{aligned} \right\} \tag{43}$$

We see that the following relations hold

$$q_{22}^\alpha = \frac{-b_3 - h_{11} q_{11}^\alpha}{h_{22}} \qquad q_{22}^\beta = \frac{-b_3 - h_{11} q_{11}^\beta}{h_{22}} \qquad \frac{q_{11}^\alpha - q_{11}^\beta}{q_{22}^\beta - q_{22}^\alpha} = \frac{h_{22}}{h_{11}} \tag{44}$$

The final result is given by

$$Q^\alpha = \begin{bmatrix} q_{11}^\alpha & q_{12} \\ q_{12} & q_{22}^\alpha \end{bmatrix} \qquad Q^\beta = \begin{bmatrix} q_{11}^\beta & q_{12} \\ q_{12} & q_{22}^\beta \end{bmatrix} \tag{45}$$

The following comments can be made about this algorithm.

When $a_0 = \det H \neq 0$ then we see from (44) that the relation between q_{11} and q_{22} is linear. The relations between q_{ii}, $i = 1, 2$ and q_{12} can be found from (42) and (43). These relations are elliptic when the solution of the inequality in Step 6 is finite. Then sets α and β are two parts of this ellipse. The relations are generally given by

$$m_{1i} q_{ii}^2 + m_{2i} q_{ii} q_{12} + m_{3i} q_{12}^2 + m_{4i} q_{ii} + m_{5i} q_{12} + m_{6i} = 0, \quad i = 1, 2 \tag{46}$$

where m_{ji}, $j = 1, 6$ is easily found from a_i, b_i, c_i, d_i and p_i. When $a_0 = \det H = 0$, then the relation between q_{ii}, $i = 1, 2$ and q_{12} is linear. This is seen from equations (40) and (41). When $a_0 = \det H < 0$, then a sign is changed and the elliptic relation above is changed to a hyperbolic relation between q_{ii}, $i = 1, 2$ and q_{12}. This is (of course) of theoretical interest because $H = H^T \geqslant 0$ when $P = P^T > 0$. See the comments in Section 6.

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This method gives the answer to the question about whether it is possible with a given control weight matrix P to determine a state weight matrix Q so that the closed loop system obtained a set of prescribed eigenvalues. This method does not just provide a yes/no answer but gives us all such matrices. When $H = H^T < 0$, then we will either have an infinite number of solutions or none, but in this case we cannot be sure that the solution of the Riccati equation is positive.

This method can be used to determine all state weight matrices Q (if there are none) that correspond to a set of prescribed eigenvalues. It is known that the behaviour of a system under feedback not only is dependent on the eigenvalues but also on the eigenvectors. As a result of this method we have the freedom to choose the corresponding eigenvectors such that some performance criteria are optimized. One such could be the quadratic performance criterion

$$\min_Q J = tr(R), \quad E(x(0)x^T(0)) = I \tag{47}$$

Another such criterion could be the Frobenius norm for the closed loop system matrix

$$\min_Q J_F = (A + BG)_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (\bar{a}_{ij})^2 \right)^{1/2} \tag{48}$$

Where $\bar{a}_{ij}$ indicates element ij in $A + BG$. The relation between this norm and the robustness in the sense of an additive uncertainty dA in the system matrix A is given in Dickman (1987). It can be shown that the robustness is maximized when the Frobenius norm for the closed-loop system is minimized. This means that we can allow larger perturbations dA when the Frobenius norm is minimized. It can also be shown that the minimization up to $\sqrt{(\sum_{i=1}^{n}(\lambda_i)^2)}$ is possible. This could happen if there are enough degrees of freedom in the controller to make $A + BG$ symmetrical. The structured singular value μ (Doyle *et al.* 1982) could also be used. A criterion which will be investigated is one which is directly connected to the eigenvalue sensitivity. Because of this algorithm it is now up to the designer to choose a suitable performance criterion.

Here we will present the equation that uniquely determines the maximum possible imaginary part of the eigenvalues in an LQ-system, irrespective of how Q are chosen. The derivation is restricted to a second order system, because the general solution is a topic of a further paper. The derivation is also restricted to the case where $a_0 = \det H \neq 0$, because there is generally no upper limit on the imaginary value when $a_0 = \det H = 0$.

If we develop the inequality (35) in terms of $b = \beta^2$ and optimize with respect to β, then after some algebra we get

$$\beta^2_{max} = b_{max} = \frac{[(h_{12}a_{22} + h_{11}a_{21}) - (h_{22}a_{12} + h_{12}a_{11})]^2}{4 \det H} \geqslant 0 \tag{49}$$

This means that the imaginary value β always is bounded by

$$0 \leqslant |\beta| \leqslant \sqrt{(b_{max})} \tag{50}$$

We see that the maximum imaginary value only is dependent on the matrices A and $H = BP^{-1}B^T$, and not on the real parts of the closed loop eigenvalues. Note that β_{max} goes to infinity when $a_0 = \det H$ goes to zero. This means that there is no upper limit on β when $a_0 = \det H = 0$. But if we restrict the state weight matrix to $Q \geqslant 0$ then there will be a maximum value. In this case β will be dependent on the real value α. This result

completes the pole placement problem, because it gives us the unique region where the eigenvalues can be located.

6. The *n*-dimensional problem

In this section we present the results from Sections 4 and 5 in a general algorithm for the determination of state weight matrix Q such that an optimal system gets a set of prescribed eigenvalues.

Algorithm 6.1

   Step 1.   Initialize $Q=Q_i$ and $R=R_i$, $i=0$. Compute $H=BP^{-1}B^T$.

   Step 2.   Compute the closed loop system matrix $A_i=A-HR$. (Equivalent to $A+BG$.) If system order $0<n\leqslant2$, jump to Step 6.

   Step 3.   Compute Real Schur Decomposition U, $T=U^TA_iU$.

   Step 4.   Specify one or two eigenvalues s_1 and s_2 to be shifted and the corresponding diagonal block T_{jj}. Reorder the RSD decomposition with an orthogonal U_p such that the block T_{jj} is placed in position T_{22} (lower right corner in T). Update the orthogonal transformation $U=UU_p$.

   Step 5.   Compute $\tilde{H}=U^THU$. Only block H_{22} in position in the lower right corner is needed.

   Step 6.   Use Algorithm 5.1 to compute Q_{22} from the decoupled sub-system given by T_{22} and H_{22}.

   Step 7.   Solve the algebraic Riccati equation (which is of a maximum order of two) for R_{22}

$$T_{22}^TR_{22}+R_{22}T_{22}-R_{22}H_{22}R_{22}+Q_{22}=0$$

   Step 8.   Back-transform solutions from

$$Q_i=U\tilde{Q}U^T \qquad R_i=U\tilde{R}U^T$$

where

$$\tilde{Q}=\begin{bmatrix}0 & 0 \\ 0 & Q_{22}\end{bmatrix} \qquad \tilde{R}=\begin{bmatrix}0 & 0 \\ 0 & R_{22}\end{bmatrix}$$

   Step 9.   Update R and the state weight matrix Q

$$Q=Q+Q_i \qquad R=R+R_i$$

   Step 10.   If all eigenvalues are shifted, then compute feedback matrix

$$G=-P^{-1}B^TR,$$

or update $A=A_i$ and $i=i+1$ and go to Step 2. The following comments can be made about this algorithm.

In Step 2 in the algorithm we suggest omitting Steps 3, 4 and 5 when the order of the system is two or less. As we have shown in Section 5, the problem can be solved directly for a second order system.

In Step 3 the RSD decomposition must be computed. We know that the computation of RSD is an intermediate step in computing the diagonal or block-

diagonal decomposition. The reordering of block T_{jj} to position T_{22} is trivial (see Golub (1983) pp. 240–242 and Stewart (1976) for details). From this we see that the RSD with the reordering of block T_{jj} is faster than block diagonalizing T, for example. Another important fact is that only to take the RSD decomposition will eliminate the problems and hazards there are to diagonalizing a defective matrix with multiple or near-multiple eigenvalues.

In Steps 6 and 7 we recommend working with a second order sub-system because the eigenvalues of the block can be moved directly. During this step we get all possible weight matrices Q, and therefore the corresponding eigenvectors can be optimized. See Section 5 for details.

We see only one Riccati equation of a maximum order of two is to be solved. This is an easy task and can be done analytically. In our algorithm we do it numerically, but it would be interesting to check if computational work and numerical precision could be saved. We recall that in the scalar case where $T_{22} = \lambda_m$, $Q_{22} = q_{nn}$ and $H_{22} = h_{nn}$, then q_{nn} can be computed from (18) and only a scalar Riccati equation has to be solved.

When the control input weight matrix $P = P^T > 0$ then $H = BP^{-1}B^T \geqslant 0$. When the system is controllable then it can be shown that the diagonal elements in H are always non-negative, and that the block H_{22} is non-negative. When the mode z_n is uncontrollable, then the nth row of $U^T B$ contains only zero elements, and the nth row and nth column in H contains only zero elements.

7. Numerical examples

7.1. Example 1

Consider the system

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} \tag{51}$$

The system eigenvalues are

$$\lambda_1(A) = -2, \quad \lambda_2(A) = -1 \tag{52}$$

The eigenvalues of the optimal closed loop system are specified as

$$s_1(A + BG) = -8, \quad s_2(A + BG) = -5 \tag{53}$$

We use Algorithm 5.1 directly and find that q_{12} must satisfy the inequality

$$J_I(q_{12}) = -0.032q_{12}^2 + 0.32q_{12} + 96.64 \geqslant 0 \tag{54}$$

The inequality is satisfied if and only if

$$-50.18 \leqslant q_{12} \leqslant 60.18 \tag{55}$$

We can now plot q_{11}, q_{22} and the measure of the performance index $J = tr(R)$ as a function of q_{12} when the closed loop system has fixed eigenvalues. This is illustrated in Fig. (1). We have also plotted the Frobenius norm $(A + BG)_F$.

For this example we have an elliptical relation between q_{ii}, $i = 1, 2$ and q_{12}. If we use $J = tr(R)$ as a measure of optimality we get one solution, where the value is $J_{min} = tr(R) = 24.52$.

$$Q^\beta = \begin{bmatrix} 62.18 & 16.03 \\ 16.03 & 109.10 \end{bmatrix} \quad R = \begin{bmatrix} 6.370 & 2.629 \\ 2.629 & 18.150 \end{bmatrix} \quad G = \begin{bmatrix} -6.370 & -2.629 \\ -0.526 & -3.630 \end{bmatrix} \tag{56}$$
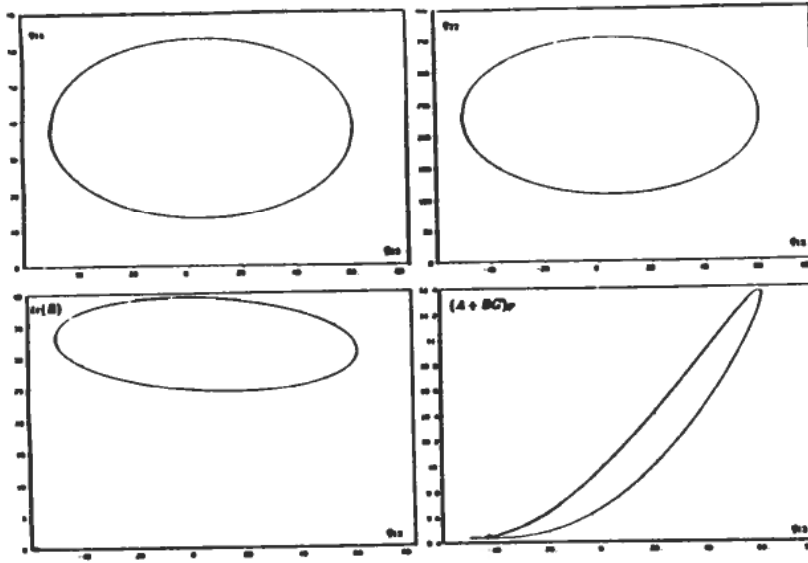
Figure 1.   Plot of q_{11}, q_{22}, $J = tr(R)$ and $(A + BG)_F$ as a function of q_{12}.

We also note that we have two diagonal solutions to the problem, namely

$$Q^\alpha = \begin{bmatrix} 13\text{-}42 & 0 \\ 0 & 352\text{-}88 \end{bmatrix} \quad Q^\beta = \begin{bmatrix} 62\text{-}58 & 0 \\ 0 & 107\text{-}12 \end{bmatrix} \tag{57}$$

The corresponding values of the performance criteria are $tr(R^\alpha) = 39\text{-}46$ and $tr(R^\beta) = 24\text{-}77$.

7.2. *Example 2*

Consider the system

$$A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \quad H = \begin{bmatrix} 4\text{-}2 & 4\text{-}6 \\ 4\text{-}6 & 5\text{-}8 \end{bmatrix} \tag{58}$$

The system eigenvalues are

$$\lambda_1(A) = -2 + i, \quad \lambda_2(A) = -2 - i \tag{59}$$

The eigenvalues of the optimal closed loop system are specified as

$$s_1(A + BG) = -8, \quad s_2(A + BG) = -5 \tag{60}$$

We use Algorithm 5.1 directly and find that q_{12} must satisfy the inequality

$$J_I(q_{12}) = -131\text{-}1q_{12}^2 - 21264\text{-}4q_{12} - 234155\text{-}5 \geqslant 0 \tag{61}$$

The inequality is satisfied if and only if

$$-150\text{-}35 \leqslant q_{12} \leqslant -11\text{-}88 \tag{62}$$

We can now plot q_{11}, q_{22} and the measure of the performance index $J = tr(R)$ as a function of q_{12} when the closed loop system has fixed eigenvalues. This is illustrated in Fig. 2. We have also plotted the Frobenius norm $(A + BG)_F$.
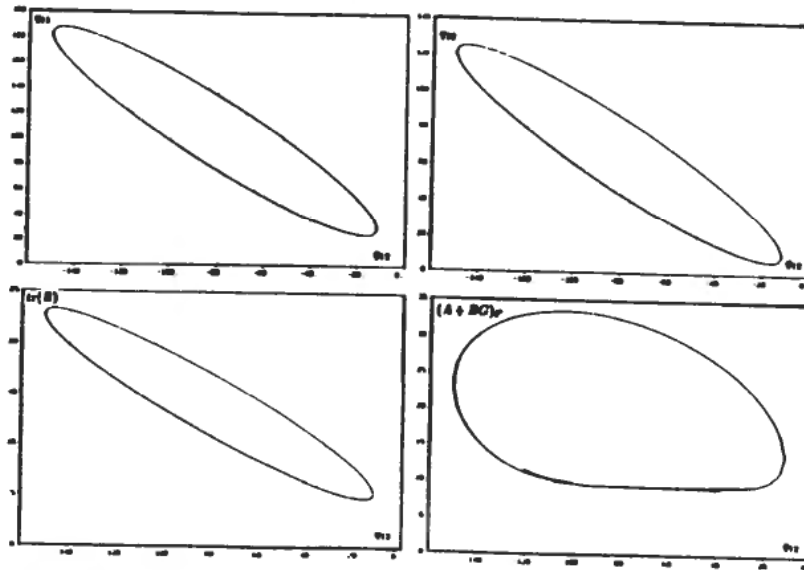
Figure 2.   Plot of q_{11}, q_{22}, $J = tr(R)$ and $(A + BG)_F$ as a function of q_{12}.

For this example we have an elliptical relation between q_{ii}, $i = 1, 2$ and q_{12}. If we use $J = tr(R)$ as a measure of optimality we get one solution, where the value is $J_{min} = tr(R) = 4.81$.

$$Q^\beta = \begin{bmatrix} 42.83 & -15.13 \\ -15.13 & 7.30 \end{bmatrix} \quad R = \begin{bmatrix} 4.056 & -1.349 \\ -1.349 & 0.755 \end{bmatrix} \quad G = \begin{bmatrix} -5.413 & 1.188 \\ -0.002 & -0.183 \end{bmatrix}$$

$$(63)$$

There are no diagonal solutions for this example.

7.3. *Example 3*

Consider the system

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad P = 1 \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{64}$$

The system eigenvalues are

$$\lambda_1(A) = 0, \quad \lambda_2(A) = 0 \tag{65}$$

The eigenvalues of the optimal closed loop system are specified as

$$s_1(A + BG) = -1, \quad s_2(A + BG) = -1 \tag{66}$$

For this system $\det(H) = 0$. We would first solve this problem by working with a scalar subsystem (Algorithm 6.1). From (17) we get $q_{22} = 1$. The solution of the scalar Riccati equation is $r = 1$.

$$Q_1 = \tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad R_1 = \tilde{R} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad G_1 = [0 \quad -1] \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \tag{67}$$

A_1 is in real Schur form. To move the other eigenvalue we must reorder the Schur form.

The reordered Schur decomposition of A_1 gives

$$T=\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad U=\begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} \quad \tilde{H}=U^T H U=\begin{bmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix} \tag{68}$$

The same procedure as above gives $q_{22}=2$, $r=2$ and

$$Q_2=U\tilde{Q}U^T=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad R_2=U\tilde{R}U^T=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad G_2=[-1 \quad -1] \tag{69}$$

The final results are given by

$$Q=Q_1+Q_2=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad R=R_1+R_2=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{70}$$

and finally

$$G=G_1+G_2=-P^{-1}B^T R=[-1 \quad -2] \quad A+BG=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \tag{71}$$

We will now solve the same problem by using Algorithm 5.1 directly. We get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_{11} \\ q_{22} \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} p_2-0\cdot q_{12}-0 \\ p_1-0\cdot q_{12}-0 \end{bmatrix}=\begin{bmatrix} p_2 \\ p_1 \end{bmatrix} \tag{72}$$

From this we get the general solution

$$Q=\begin{bmatrix} 1 & c \\ c & 2 \end{bmatrix}, \quad c\in\mathbf{R}^1 \tag{73}$$

We see that there are an infinite number of solutions, but $R=R^T>0$ is only guaranteed when $Q>0$. This result was not found from the first procedure.

7.4. Example 4

Consider the system

$$A=\begin{bmatrix} -6 & 5 \\ 5 & -6 \end{bmatrix} \quad H=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{74}$$

The system eigenvalues are

$$\lambda_1(A)=-11, \quad \lambda_2(A)=-1 \tag{75}$$

The eigenvalues of the optimal closed loop system are specified as

$$s_1(A+BG)=-14, \quad s_2(A+BG)=-14 \tag{76}$$

We use Algorithm 5.1 directly and find that q_{12} must satisfy the inequality

$$J_t(q_{12})=-4q_{12}^2+480q_{12}+14400\geqslant 0 \tag{77}$$

The inequality is satisfied only for $q_{12}=60$. For this value we get the only solution

$$Q=\begin{bmatrix} 135 & 60 \\ 60 & 135 \end{bmatrix} \tag{78}$$

8. Concluding remarks

1. We have discussed a new method for solving the linear quadratic pole placement problem in detail. The method is numerically stable and has considerable advantages over the traditional diagonalization approach. Two eigenvalues can be moved to new real or complex locations in each step, and only a Riccati equation of an order of not more than two has to be solved. The final solution to the Riccati equation for the n-dimensional problem is guaranteed to be symmetrical if the two dimensional Riccati equation is solved symmetrically. Note that a symmetrical solution is not guaranteed when using Laub's (1979) method.

2. The new method gives the answer to the question about whether it is possible to determine a state weight matrix Q with a given control weight matrix P so that the closed loop system obtains a set of prescribed eigenvalues. This method does not only provide a yes/no answer but also gives us all such matrices (for a second order system). It could be an infinite number, a constrained space or none.

3. For a second order (sub) system we have presented the general relations between the state weight matrices Q and the closed loop eigenvalues. From this result we have been able to derive the maximum possible imaginary part of the eigenvalues in a LQ-optimal system, irrespective of how the state weight matrix Q is chosen.

4. This method can be used to determine all state weight matrices Q (if there are any) that correspond to a set of prescribed eigenvalues (for a second order system). It is known that the behaviour of a system under feedback not only is dependent on the eigenvalues but also on the eigenvectors. As a result of this method we have the freedom to choose the corresponding eigenvectors such that some performance criteria are optimized.

5. This method may be used in the same way to design optimal estimators with prescribed eigenvalues. The method can also be used on discrete systems.

6. The implementation of the algorithm is simple. We recommend the use of sequence ORTHES and ORTRAN to reduce a general matrix by orthogonal similarity transformations to upper Hessenberg form. These routines are all available in EISPACK (Smith 1976). The reduction to real Schur form is done by Stewart's (1976) HQR3 software. The reordering of block T_{jj} (corresponding to the eigenvalues which are to be moved) to position T_{22} is done by the application of the HQR3 routine EXCHNG.

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Appendix

The coefficients in equations (22) and (23) are given below

$$a_0 = h_{11}h_{22} - h_{12}^2 = \det H \tag{79}$$

$$a_1 = h_{11}t_{22}^2 - 2h_{12}t_{12}t_{22} + h_{22}t_{12}^2 \tag{80}$$

$$a_2 = h_{11}t_{21}^2 - 2h_{12}t_{11}t_{21} + h_{22}t_{11}^2 \tag{81}$$

$$b_1 = h_{11} \tag{82}$$

$$b_2 = h_{22} \tag{83}$$

$$d_1 = 2h_{12} \tag{84}$$

$$d_2 = t_{11}^2 + t_{22}^2 + 2t_{12}t_{21} = tr(T^2) \tag{85}$$

$$k_1 = 2(h_{12}t_{21} - h_{22}t_{11})t_{12} + 2(-h_{11}t_{21} + h_{12}t_{11})t_{22} \tag{86}$$

$$k_2 = (t_{11}t_{22} - t_{12}t_{21})^2 = (\det T)^2 \tag{87}$$

The coefficients in the inequality (35) that constrains q_{12} is given below. Only c_2 and c_3 are dependent on the location of the new eigenvalues

$$c_1 = -4a_0^3 = -4(\det H)^3 \tag{88}$$

$$c_2 = g_1 p_1 + g_2 \tag{89}$$

$$c_3 = f_1 p_1^2 + f_2 p_1 + f_3 + f_4 p_2 \tag{90}$$

where the coefficients g_i and f_i are only dependent on the elements in matrices T and H.

$$g_1 = -2a_0^2 d_1 \tag{91}$$

$$g_2 = a_0(4b_1 b_2 k_1 + 2d_1(a_0 d_2 - a_1 b_2 - a_2 b_1)) \tag{92}$$

$$f_1 = a_0^2 \tag{93}$$

$$f_2 = -2a_0(a_0 d_2 - a_1 b_2 - a_2 b_1) \tag{94}$$

$$f_3 = (a_0 d_2 - a_1 b_2 - a_2 b_1)^2 + 4b_1 b_2(a_0 k_2 - a_1 a_2) \tag{95}$$

$$f_4 = -4a_0 b_1 b_2 \tag{96}$$

The coefficients in equations (40) and (41) are given by

$$v_1 = \frac{-b_2 k_1 + a_2 d_1}{a_1 b_2 - a_2 b_1} \tag{97}$$

$$v_2 = \frac{b_2(p_2 - k_2) - a_2(p_1 - d_2)}{a_1 b_2 - a_2 b_1} \tag{98}$$

$$w_1 = \frac{b_1 k_1 - a_1 d_1}{a_1 b_2 - a_2 b_1} \tag{99}$$

$$w_2 = \frac{-b_1(p_2 - k_2) + a_1(p_1 - d_2)}{a_1 b_2 - a_2 b_1} \tag{100}$$

The Schur formula is given by

$$\det \begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det A \det [B - CA^{-1}D] \tag{101}$$