MODELING, IDENTIFICATION AND CONTROL, 1989, VOL. 10, NO. 3, 125–134
doi:10.4173/mic.1989.3.1

Performance evaluation of positive regulators for population control†

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Keywords: *Pole assignment, industrial regulators, Leslie systems, population control, simulation.*

This paper deals with real time control of age-structured populations described by Leslie models with positive inputs. The classical industrial and pole-assignment regulators are adapted to this class of positive systems and their performance is evaluated through simulation. The influence of noise on cost and robustness of the controlled system and the role of the information structure are discussed in some detail.

1. Introduction

The problem of real time control of age-structured populations (Luenberger (1969)) is considered in this paper. The population is described by a classical Leslie model (Leslie 1945) a positive discrete time linear system of the form $x(t+1) = Ax(t) + bu(t)$. Each component $x_i(t)$ of the n-dimensional state vector $x(t)$ represents the number of individuals of age i present in the system in year t, while the non-negative control variable $u(t)$ is a measure of the stocking rate. The non-zero elements of the matrix A are the survival and fertility coefficients of each age class, while each component b_i of the vector b represents the number of i-year old individuals entering into the system for one unit of control.

The stability properties of these systems were investigated many years ago (Leslie (1945), Sykes (1969)), while reachability and stabilization via linear feedback were studied by two of the authors (Muratori and Rinaldi (1987, 1988)). Under the assumption that the population vanishes when it is not sustained by stocking, the analysis shows that any equilibrium state x^* can be reached in finite time from any initial condition and that the input sequence $u(0)$, $u(1)$, ... which performs the operation can be locally computed by a pole assignment regulator. In this paper we extend the analysis and determine the performance of this and other control laws in real situations, namely for noisy population dynamics and large deviations from the equilibrium. Moreover, we assume that the population is observed only through a single-value indicator $y(t)$ (e.g. total population, number of newly-born individuals, number of deaths, ...). Thus, given the information $y(t)$, the problem is to specify a positive control $u(t)$ in such a way that the desired population levels are maintained in spite of the numerous causes of uncertainty (observation errors and fluctuations of survival and fertility coefficients due to variations of environmental conditions).

Received 1 November 1988.

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Of course, different control policies can be used for this purpose, among which we consider the open loop control scheme, the PID regulator adapted to the case $u(t) \geqslant 0$, and the pole assignment regulator which computes the stocking rate on the basis of a real time estimate of the number of individuals of each age class. The performance of each stocking policy is evaluated (through extensive simulation) for different types of populations, for the most common observation schemes, and for various intensities of measurement and process noise. The performances of the regulators are then compared by means of classical cost and robustness indicators and some general conclusions are drawn from these comparisons.

2. Background on deterministic population control

A *Leslie system with positive inputs* is a single-input n-dimensional discrete-time dynamic system

$$x(t+1) = Ax(t) + bu(t) \tag{1}$$

with

$$A = \begin{vmatrix} s_0 f_1 & s_0 f_2 & \dots & s_0 f_{n-1} & s_0 f_n \\ s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & s_{n-1} & 0 \end{vmatrix} \qquad b = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{vmatrix} \tag{2}$$

and $f_i \geqslant 0$, $0 < s_i \leqslant 1$, $b_i \geqslant 0$, $x_i(0) \geqslant 0$ $\forall i$ and $u(t) \geqslant 0$ $\forall t$. System (1–2) is a positive dynamic system since $x_i(t) \geqslant 0$ $\forall (i, t)$.

Systems of this kind, with $b_i = 0$ $\forall i$, have been extensively used in recent decades to describe the explosion or extinction of isolated populations in plant and animal ecology, cell biology, and demography. The linear approach to population analysis is credited to Leslie and the matrix A is often called a *Leslie matrix*. The ith component of the state vector represents the number of individuals of age i in year t before the reproduction season; the positive parameter s_i is the *survival* rate of the ith age group during one year, while the non-negative *fertility* coefficient f_i is the number of offspring per year per member of age group i. The control variable $u(t)$ is a measure of the *stocking* rate and b_i is the number of individuals of age i stocked into the system per unit of control.

In order to avoid the analysis of trivial cases, we assume that at least some of the individuals which are stocked into the system have the chance to reproduce before dying, i.e.

$$b_j f_k > 0 \quad \text{for some} \quad (j, k) \quad \text{with} \quad j \leqslant k. \tag{3}$$

The coefficients of the characteristic polynomial

$$\Delta_A(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

are given by

$$a_i = -s_0 \cdot s_1 \dots s_{i-1} f_i \leqslant 0 \qquad i = 1, \dots, n$$

and at least one of them is negative (see Eqn. (3)). This implies the existence and uniqueness of a positive eigenvalue $\hat{\lambda}$ which is the unique dominant eigenvalue (Frobenius eigenvalue), if and only if (Sykes (1969)), the largest common divisor of the subscript js for which $f_j > 0$ is equal to one, a condition which is always satisfied in applications of practical interest.

The stability of system (1, 2) is strictly related to the existence of positive equilibria (u^*, x^*) and their reachability. In fact, the following properties hold (Muratori and Rinaldi (1987)).

Property 1. System (1, 2) is asymptotically stable if and only if its non-trivial equilibria (u^*, x^*) are strictly positive.

Property 2. In system (1, 2) with $u(t) \geqslant 0$ the set $X_t(x(0))$ of states reachable from an initial state $x(0)$ in $t \geqslant n$ steps is given by

$$X_t(x(0)) = \{x \mid x = A^t x(0) + C^+[b, Ab, \dots, A^{n-1}b]\}$$

where $C^+[b, Ab, \dots, A^{n-1}b]$ is the positive cone containing all the non-negative linear combinations of the n reachability vectors $b, Ab, \dots, A^{n-1}b$.

Property 3. If system (1, 2) is asymptotically stable its non-trivial equilibria x^* belong to $C^+[b, Ab, \dots, A^{n-1}b]$.

Properties 2 and 3 and the fact that $A^t x(0)$ tends towards zero in a stable system imply $x^* \in X_t(x(0))$ for a sufficiently large t. This means that any equilibrium x^* can be reached in finite time from any initial state $x(0)$. Unfortunately, the proof of this result (Muratori and Rinaldi (1987)) is not constructive, so that the problem of determining the control sequence $u(0), u(1), \dots, u(t-1)$ which guarantees that $x(t) = x^*$ is not yet solved. Nevertheless, the problem has a local solution if the reachability vectors $b, Ab, \dots, A^{n-1}b$ are linearly independent. In fact, in this case one can consider a linear control law of the form

$$u(t) = u^* + k^{\mathrm{T}}(x(t) - x^*) \tag{4}$$

and determine the vector k (pole assignment problem) in such a way that the matrix $(A + bk^{\mathrm{T}})$ of the closed loop system

$$x(t+1) = (A + bk^{\mathrm{T}})(x(t) + bu^* - bk^{\mathrm{T}}x^*$$

has all its eigenvalues equal to zero. Thus, if $x(0)$ is sufficiently close to x^*, the value $u(0)$ suggested by (4) is positive and hence feasible (as well as all subsequent control values $u(1), u(2), \dots$), and the state of the closed loop system reaches x^* in, at most, n steps.

3. Design of heuristic positive regulators

We now assume that the population is observed through a single-value indicator $y(t)$ which is positively and linearly related to the state vector $x(t)$, i.e.

$$y(t) = c^{\mathrm{T}}x(t)\ c_i \geqslant 0, \qquad i = 1, \dots, n \tag{5}$$

For example, $y(t)$ might be

(i) total number of individuals at the beginning of the year ($c_i = 1, i = 1, \dots, n$).

(ii) total number of newly-born individuals at the end of the reproduction season ($c_i = f_i, i = 1, \dots, n$).

(iii) total number of newly-born individuals at the end of the year ($c_i = s_0 f_i$, $i = 1, \dots, n$).

(iv) total number of individuals dying during the year ($c_i = f_i(1 - s_0) + (1 - s_i)$, $i = 1, \dots, n$).

Therefore, in the absence of noise the system is described by the linear equations (1, 5) in which (A, b, c^T) are non-negative, as well as the current and reference control, state and output values $(u(t), x(t), y(t))$ and (u^*, x^*, y^*). Feasible regulators for this class of systems, called *positive regulators*, must be such that the control value $u(t)$ is non-negative, no matter what the past and present values of input and output are. In the following we consider three different positive regulators.

Open-loop control (OL)

This control scheme, defined by

$$u(t) = u^*,$$

is considered because it is often used in practice (for example, by fishing agencies releasing juveniles in streams).

Industrial regulator (PID)

The classical PID linear regulator

$$\tilde{u}(t) = u^* + \alpha_P(y(t) - y^*) + \alpha_I w(t) + \alpha_D z(t) \tag{6a}$$

$$w(t+1) = \beta w(t) + y(t+1) - y^* \tag{6b}$$

$$z(t+1) = y(t+1) - y(t) \tag{6c}$$

identified by the four parameters $(\alpha_P, \alpha_I, \alpha_D, \beta)$ can be transformed into a positive PID regulator by simply adding the non-linear rule

$$u(t) = \max\{0; \tilde{u}(t)\}. \tag{7}$$

In the following the PID regulator (6, 7) is designed by minimizing the modulus of the dominant eigenvalue of the linear closed loop system (1, 5, 6) with $u = \tilde{u}$. This can simply be done by coupling a general routine for the computation of the $(n + 2)$ eigenvalues $\lambda_1, \dots, \lambda_{n+2}$ of the state-transition matrix

$$A_{PID} = \begin{vmatrix} (A + \alpha_P bc^T) & \alpha_I b & \alpha_D b \\ c^T(A + \alpha_P bc^T) & (\beta + \alpha_I c^T b) & \alpha_D c^T b \\ c^T(A + \alpha_P bc^T - I) & \alpha_I c^T b & \alpha_D c^T b \end{vmatrix}$$

of the closed-loop system (1, 5, 6) with a gradient method solving the four-dimensional problem

$$|\lambda_{\max}| = \min_{(\alpha_P, \alpha_I, \alpha_D, \beta)} \max_i \{|\lambda_i(\alpha_P, \alpha_I, \alpha_D, \beta)|\}. \tag{8}$$

Pole assignment regulator (PA)

The classical pole-assignment regulator, constituted by a state reconstructor

$$\hat{x}(t+1|t+1) = \hat{x}(t+1|t) + l(\hat{y}(t+1|t) - y(t+1)) \tag{9a}$$

$$\hat{x}(t+1|t) = A\hat{x}(t|t) + bu(t) \tag{9b}$$

$$\hat{y}(t+1|t) = c^T\hat{x}(t+1|t) \tag{9c}$$

and a linear algebraic controller

$$u(t) = u^* + k^T(\hat{x}(t \mid t) - x^*) \tag{10}$$

can be transformed into a positive regulator by substituting the linear control law (10) with the non-linear control law

$$u(t) = \max \{0; u^* + k^T(\tilde{x}(t \mid t) - x^*)\} \tag{11a}$$

where

$$\tilde{x}_i(t \mid t) = \max \{0; \hat{x}_i(t \mid t)\} \qquad i = 1, \ldots, n \tag{11b}$$

Since the triplet (A, b, c^T) is, in general, completely reachable and completely observable in the classical sense (Muratori and Rinaldi (1987)) and A is non-singular, the eigenvalues of the matrices $(A + bk^T)$ and $(A + lc^TA)$ can be assigned at will (Luenberger (1969)) by properly selecting the parameters $k_1, \ldots, k_n$ of the controller and the parameters $l_1, \ldots, l_n$ of the state reconstructor. In particular, if these eigenvalues are forced to be equal to zero the regulated system (1, 5, 9, 11) in the absence of noise converges to x^* in at most $2n$ steps provided $x(0)$ and $\hat{x}(0)$ are sufficiently close to x^*.

4. Sources of uncertainty, robustness and cost of control

Animal and plant populations are in general influenced by a large number of environmental factors such as food abundance, meteorological conditions and predator's density. These factors are quite independent from each other and vary randomly in time. The sources of uncertainty can be modelled by introducing suitable multiplicative noise terms in Eqns. (1, 2) which represent the reproduction and growth processes and in Eqn. (5). The assumptions on the noise (white and log-normal) made in this paper are fairly standard and in agreement with biological evidence.

Survival

The survival at birth (s_0 in Eqn. (2)) is very frequently the most important source of uncertainty in the system. There are obvious reasons for this and statistical evidence in field data. This is why s_0 has been considered the only fluctuating parameter in the Leslie matrix in some studies (Ginzburg *et al.* 1984, O'Neil *et al.* 1981) on age structured uncontrolled populations ($b = 0$ in Eqn. (1)). In general the survival coefficient s_0 is a lumped description of a number of factors which characterize a sequence of short periods immediately after birth. If these periods are p and each one of them is characterized by a length r_i and a varying mortality rate m_i^t we can write

$$s_0^t = \exp\left[-\sum_{i=1}^{p} m_i^t r_i\right] \tag{12}$$

Since the mortality rates m_i^t are, to a large extent, independent random variables with mean m_i and standard deviation s_i, the variable

$$m_0^t = \sum_{i=1}^{p} m_i^t r_i \tag{13}$$

has a mean value $m_0 = \sum m_i r_i$ and variance $s_0^2 = \sum s_i^2 r_i^2$. Moreover, if p is large and s_i/m_i is roughly the same for all mortality factors the random variable (13) is normally distributed, i.e.

$$m_0^t = N(m_0, s_0)$$

and s_0 is proportional to m_0. Thus, it follows from Eqn. (12) that the survival coefficient can be expressed as

$$s_0^t = e_0^t s_0 \tag{14}$$

where $s_0 = \exp[-m_0]$ and e_0^t is distributed log-normally with standard deviation proportional to $[-\ln s_0]$.

For the same reasons above we can assume that the survival coefficients s_i^t, $i = 1, \ldots, n-1$ are also the product of a log-normal variable times a standard value s_i, but we can also assume that the log-normal variable is the same for all age classes because all adults are in general subject to the same environmental stresses at the same time. Of course, the standard deviation of the log-normal noise must be consistent with the nominal values s_i of the survival coefficients.

Fertility

The fertility coefficients vary in time with respect to their nominal values f_i, $i = 1, \ldots, n$, i.e.

$$f_i^t = e_f^t f_i$$

where e_f^t is assumed to be a white stationary process with log-normal distribution. The noise does not depend upon i because the relative variations of fertility mainly depend upon meteorological conditions and food abundance in very particular periods of the year and are therefore independent of the age of the individuals.

Measurements

The measure of the indicator $y(t)$ of the population is of course affected by noise. Since $y(t)$ is a positive variable we assume that

$$y(t) = e_y^t c^T x(t)$$

where e_y^t is a white stationary process with log-normal distribution. Notice that in some cases the vector c^T is also affected by noise because its elements c_i depend upon f_i and/or s_i.

In order to compare the performances of the different regulators we need meaningful indicators of robustness, and cost of control which must be evaluated through simulations of the regulated system in noisy conditions. This is because explicit formulas for the computation of such indicators are not known when the elements of the matrices (A, b, c^T) of the system are perturbed by white noise.

Robustness

Since the aim of the regulation is to keep the population as close as possible to x^*, the robustness of the controlled system with respect to noise can be judged by means of the following *noise to signal ratio*

$$J_1 = E\left[\frac{1}{n}\sum_{i=1}^{n} |x_i(t) - x_i^*|/x_i^*\right]$$

where $E[\,\cdot\,]$ denotes expected value. Notice that this indicator is equally sensitive to positive and negative errors of all age classes.

Cost of control

In many applications the control action is associated with a cost which is roughly proportional to the stocking rate $u(t)$. For this reason the cost of the control is measured by the normalized *mean value of the control variable*

$$J_2 = E[u(t)/u^*].$$

5. Simulation and performance evaluation

The dynamic behaviour of the controlled system has been simulated for eight populations with maximum age $n = 10$ in order to evaluate and compare the efficiency of the three regulators described in § 3. The survival and fertility coefficients of each population are reported in Table 1 where rows (1, 2), (3, 4), (5, 6), and (7, 8) refer to the same animals (a fish, a bird, a deer, and a squirrel). The populations are all stable ($\hat{\lambda} < 1$) because they correspond either to a hypothetical exploitation (odd rows) which lowers the survival coefficients with respect to natural conditions or to a hypothetical environmental degradation (even rows) which lowers the fertility coefficients.

Each system has been simulated for at least 50 generations (500 steps) and for eight levels of noise intensity in order to obtain reliable estimates of J_1 and J_2.

s_0 / f_1	s_1 / f_2	s_2 / f_3	s_3 / f_4	s_4 / f_5	s_5 / f_6	s_6 / f_7	s_7 / f_8	s_8 / f_9	s_9 / f_{10}	$\hat{\lambda}$
0.006	45	27	26	26	25	25	25	25	25	
—	5	11	18	24	31	34	41	45	46	0·65
0·006	70	55	53	52	51	51	50	50	50	
—	0·5	1·1	1·8	2·4	3·1	3·4	4·1	4·5	4·6	0·64
50	80	36	37	38	39	39	38	38	37	
—	0·40	0·45	0·50	0·50	0·50	0·50	0·50	0·50	0·50	0·65
50	80	90	93	96	98	97	96	95	94	
—	0·020	0·023	0·025	0·025	0·025	0·025	0·025	0·025	0·025	0·68
70	92	48	49	48	42	28	25	22	20	
—	—	0·10	0·40	0·50	0·50	0·50	0·50	0·45	0·40	0·66
70	92	96	98	97	84	55	50	45	40	
—	—	0·01	0·04	0·05	0·05	0·05	0·05	0·04	0·04	0·69
40	24	30	33	34	33	30	28	24	27	
0·6	1·2	1·9	1·9	1·9	1·9	1·9	1·9	1·8	1·6	0·66
40	55	68	75	79	77	70	65	56	40	
0·06	0·12	0·19	0·19	0·19	0·19	0·19	0·19	0·18	0·16	0·67

* Fertility coefficients f_i, $i = 1, \ldots, n$, are in thousands.

Table 1. Survival and fertility coefficients (and Frobenius eigenvalue $\hat{\lambda}$) of 8 populations used for simulation. Survival coefficients are expressed as percentages.

Moreover, the simulations have been performed for the four measurement schemes indicated by (i), (ii), (iii), and (iv) in § 3. On the contrary, only one type of control has been considered, namely that corresponding to stocking with individuals of the first fertile age class. This means that the components b_i of the vector b are all zero but one (the first for populations *n.* 7, 8, the second for populations *n.* 1, 2, 3, 4 and the third for populations *n.* 5, 6 (see Table 1)). For each regulator the four different noise sources (survival at birth (s_0), adult survival (s_i, $i=1,\ldots,n-1$), fertility (f_i, $i=1,\ldots,n$), and measurement (y) have been considered as acting separately or at the same time, for a total of 4096 simulations. The results of the analysis are reported in the following tables.

(*a*): J_1

	s_0	s_i	f_i	y
OL	13	13	6	100
PID	54	–	87	–
PA	4	15	3	–

(*b*): J_2

	s_0	s_i	f_i	y
OL	14	–	–	–
PID	13	2	41	34
PA	7	1	13	35

(*c*): J_1 and J_2

	s_0	s_i	f_i	y
OL	2	–	–	–
PID	9	–	37	–
PA	1	–	2	–

Table 2. Rows refer to open loop (OL) control, PID regulator and pole assignment (PA) regulator. Columns refer to independent sources of uncertainty (see § 4). Entry (i, j) is the percentage of cases in which regulator i is at least one per cent better than the others when only noise j is present. In Table 2(*a*) and 2(*b*) the comparison is made with respect to robustness (J_1) and cost of control (J_2), while Table 2(*c*) represents a global comparison.

Table 2(*a*) shows the percentage of cases in which each regulator is more robust than the two others for each single noise source. For example, the first entry equal to 13 means that, if the survival at birth (s_0) is the only source of uncertainty, in 13% of the cases the open loop (OL) control scheme has a noise to signal ratio J_1 lower than those obtained with the PID regulator and the pole assignment (PA) regulator. The sum of the figures in each column is often lower than 100 because only the cases in which the values of the indicators differ at least of one per cent have been considered as meaningful in the statistics. In a similar way Table 2(*b*) makes reference to the cost of control J_2 while Table 2(*c*) reports the percentage of cases in which each regulator dominates the two others (i.e. is better than the two others from both points of view (robustness and cost of control)).

Table 2(*a*) shows that the PID regulator is the most robust one when the sources of uncertainty are the survival at birth (s_0) or the fertilities (f_i). On the contrary, if the noisy parameters are the survival coefficients of the adults (s_i) the PID regulator is less robust. Finally, if the system is deterministic but there are measurement errors (fourth column) the open loop strategy is obviously the best. In any case, Table 2(*a*) points out that the open loop control scheme is not as bad as one could imagine and that the PA regulator does not perform as well as the PID regulator. As far as the cost of control is concerned (see Table 2(*b*)), the PID and the PA regulators are in general better than the open loop control and still the PID has a detectable advantage when the noisy parameters are s_0 and f_i. Table 2(*c*) also indicates that these are the only cases in which one regulator (the PID) dominates the others.

Table 3 compares the three regulators and the four different measurement schemes (i), (ii), (iii) and (iv) in the case where all the noise sources are active at the same time. Table 3(*a*) shows that the PID is definitely more robust than the two

	(i)	(ii)	(iii)	(iv)
OL	14	16	13	53
PID	58	75	86	20
PA	–	3	–	5

(*a*): J_1

	(i)	(ii)	(iii)	(iv)
OL	2	–	5	–
PID	9	23	16	20
PA	16	3	16	25

(*b*): J_2

	(i)	(ii)	(iii)	(iv)
OL	–	–	–	–
PID	3	19	13	14
PA	–	–	–	–

(*c*): J_1 and J_2

Table 3. Rows refer to open loop (OL) control, PID regulator and pole assignment (PA) regulator. Columns refer to the four measurement schemes described in § 3. Entry (i, j) is the percentage of cases in which regulator i is at least one per cent better than the others when the jth measurement scheme is used. In Tables 3(*a*) and 3(*b*) the comparison is made with respect to robustness (J_1) and cost of control (J_2), while Table 3(*c*) represents a global comparison.

other regulators with the exception of case (iv) in which the output observations are the number of individuals dying during the year (in such a case the open loop control turns out to be the most robust scheme). Table 3(*b*) shows that the PA regulator gives rise to a lower control cost than the PID when the total number of individuals is known (case (i)) while the opposite is true when the output y is the total number of newly-born individuals at the end of the reproduction season.

Table 3(*c*) indicates that the PID regulator is the only one which has the chance to dominate the others, but this chance is particularly low when the population indicator y is the total number of individuals.

6. Concluding remarks

In this paper we have compared the performance of three different control schemes (open loop control, PID regulator and pole assignment regulator) which can be used to determine each year the number of individuals to be stocked in a stable age-structured system. The analysis has been carried out by making realistic assumptions on the variability of the parameters which characterize the population. This corresponds to determine the average value of the control variable and the noise to signal ratio of the state of a discrete-time nonlinear system (constituted by a linear system (A, b, c^T) and a nonlinear regulator) in which some parameters of the matrices (A, b, c^T) are discrete white noise. Since no theoretical results are known for such a problem, the analysis has been carried out through extensive simulation.

The results obtained in the paper are the first known on this subject and certainly need to be further detailed. In particular, it would be interesting to determine the control cost. Moreover, the open loop control scheme is not as bad as one might imagine.

The results obtained in the paper are the first known on this subject and certainly need to be further detailed. In particular, it would be interesting to determine the performance of the regulated system when the PID and the pole assignment regulators are not designed with the simple (but "extreme") min–max criterion used in this paper (see § 3). A further area would be exploring the possibility of using more complex control strategies.

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