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Trajectory generation for manipulators using linear quadratic optimal tracking

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The reference trajectory is normally known in advance in manipulator control which makes it possible to apply linear quadratic optimal tracking. This gives a control system which rounds corners and generates optimal feedforward. The method may be used for references consisting of straight-line segments as an alternative to the two-step method of using splines to smooth the reference and then applying feedforward. In addition, the method can be used for more complex trajectories. The actual dynamics of the manipulator are taken into account, and this results in smooth and accurate tracking. The method has been applied in combination with the computed torque technique and excellent performance was demonstrated in a simulation study. The method has also been applied experimentally to an industrial spray-painting robot where a saw-tooth reference was tracked. The corner was rounded extremely well, and the steady-state tracking error was eliminated by the optimal feedforward.

1. Introduction

In several applications of robotic manipulators, the reference trajectory is specified in terms of straight-line segments. The corners of this trajectory are usually rounded to avoid jerky motion of the manipulator. This is normally done by using a spline approximation consisting of straight lines joined by polynomials (Paul (1981) and Craig (1986)). The modified trajectory is then used to generate feed-forward to the manipulator.

This method is difficult to apply when the straight-line segments are too short to be splined together without exceeding the acceleration capabilities of the manipulator. Further, the method becomes complicated when more general reference trajectories are used.

We have therefore used linear quadratic optimal control theory instead of the purely kinematic splining method. The reference trajectory is known, which makes it possible to use optimal feedforward from future references (Athans and Falb (1966)). This method rounds the corner and at the same time feedforward is generated. The actual dynamics of the manipulator are taken into account, and this results in smooth and accurate tracking.

This method is purely off-line, so there are no problems with on-line computational requirements. The linear quadratic optimal control scheme was applied to a manipulator in combination with the computed torque technique (Bejczy (1974)) in a simulation experiment. The results are presented in this paper together with

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experimental results from an implementation on the Trallfa TR400, a hydraulic spray-painting robot. This robot had independent joint controllers as load pressure sensors are required to apply the computed torque technique (Egeland (1987)).

2. Dynamic model

The equation of motion for a general n-link manipulator can be found from Newton-Euler's equations (Luh, Walker and Paul 1980), this is written

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\tau} \tag{1}$$

where $\boldsymbol{q}$ is the vector of joint coordinates, $M(q)$ is the inertia matrix, $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a vector consisting friction, gravity, Coriolis and centrifugal terms, and $\boldsymbol{\tau}$ is the vector of input generalized forces.

It is well-known that when current-controlled DC motors are used, the computed torque technique (Athans and Falb 1966) may be applied, where

$$\boldsymbol{\tau} = M(\boldsymbol{q})\boldsymbol{u} - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{2}$$

This give the equivalent model

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 \tag{3}$$

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{u} \tag{4}$$

where $\boldsymbol{x}_1 = \boldsymbol{q}, \boldsymbol{x}_2 = \dot{\boldsymbol{q}}$ and $\boldsymbol{u}$ is the transformed control vector.

A model of the same structure as (1) can be obtained when voltage-controlled DC motors are used, and this makes it possible to apply the computed torque technique for this type of actuator too.

When hydraulic actuators are used, one additional state must be added for each joint to model the manipulator with actuators (Merritt (1967)). When the load pressure is used as the additional state, the dynamical model consists of (1) and

$$\dot{\boldsymbol{p}}_L = C\boldsymbol{p}_L + D\dot{\boldsymbol{q}} + B\boldsymbol{u}_v \tag{5}$$

$$\boldsymbol{\tau} = D_m \boldsymbol{p}_L \tag{6}$$

where $\boldsymbol{p}_L$ is the vector of load pressures, $\boldsymbol{u}_v$ is the vector of value control signals and $C, D, D_m = \text{diag}\,\{D_{mi}\}$ and $B = \text{diag}\,\{K_{vi}\}$ are constant diagonal matrices.

The computed torque technique can be applied to a hydraulic manipulator using load pressure feedback in an inner loop (Egeland (1987)). However, when the inertia coupling is not too strong, it is hard to justify the cost of the additional load pressure sensors. In this case, independent joint control may be the best solution. Joint i is then described by

$$\dot{z}_1 = z_2 \tag{7}$$

$$\dot{z}_2 = z_3 \tag{8}$$

$$\dot{z}_3 = -\omega_h^2 z_2 - 2\zeta\omega_h z_3 + K\omega_h^2 v \tag{9}$$

where $z_1 = q_i, z_2 = \dot{q}_i, z_3 = \ddot{q}_i$ and $v = u_{vi}$ ω_h is the undamped resonant frequency, ζ the relative damping and $K = K_{vi}/D_{mi}$. The relative damping is usually in the interval $0{\cdot}2 < \zeta < 0{\cdot}7$.

3. Review of the linear quadratic tracking problem

A reference $\boldsymbol{r}(t)$, $t_0 \leqslant t \leqslant t_f$, is given to the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{10}$$

$$\boldsymbol{y} = D\boldsymbol{x} \tag{11}$$

Accurate tracking without excessive control energy can be achieved using a linear quadratic performance index:

$$J = \tfrac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t_f)S\boldsymbol{e}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} (\boldsymbol{e}^{\mathrm{T}}Q\boldsymbol{e} + \boldsymbol{u}^{\mathrm{T}}P\boldsymbol{u})\,dt \tag{12}$$

where $\boldsymbol{e} = \boldsymbol{r} - \boldsymbol{y}$, S and Q are positive semidefinite matrices and P is a positive definite matrix.

The optimal control is then (Athans and Falb (1966))

$$\boldsymbol{u} = G\boldsymbol{x} - P^{-1}B^{\mathrm{T}}\boldsymbol{h} \tag{13}$$

where

$$G = -P^{-1}B^{\mathrm{T}}R \tag{14}$$

R is the positive definite solution of the Riccati equation

$$R = -RA - A^{\mathrm{T}}R + RBP^{-1}B^{\mathrm{T}}R - D^{\mathrm{T}}QD \tag{15}$$

with the boundary condition

$$R(t_f) = D^{\mathrm{T}}SD \tag{16}$$

$\boldsymbol{h}$ is found from

$$\dot{\boldsymbol{h}} = -(A + BG)^{\mathrm{T}}\boldsymbol{h} + D^{\mathrm{T}}Q\boldsymbol{r} \tag{17}$$

with the boundary condition

$$\boldsymbol{h}(t_f) = -D^{\mathrm{T}}S\boldsymbol{r}(t_f) \tag{18}$$

The optimal trajectory is the solution of the linear differential equation

$$\dot{\boldsymbol{x}} = (A + BG)\boldsymbol{x} - BP^{-1}B^{\mathrm{T}}\boldsymbol{h} \tag{19}$$

System (17) is the adjoint of (19).

$\boldsymbol{h}$ is found by solving (17) backwards in time from t_f. This can be done off-line when $\boldsymbol{r}(t)$ is known in advance for $t_0 \leqslant t < t_f$.

It is seen from (13) that the controller uses information about the future reference values.

4. Linear quadratic tracking and the computed torque technique

The computed torque technique is widely used in manipulator control. The optimal control is easily found when the performance functional is

$$J = \tfrac{1}{2}\Delta\boldsymbol{q}^{\mathrm{T}}(t_f)S\Delta\boldsymbol{q}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} (\Delta\boldsymbol{q}^{\mathrm{T}}Q\Delta\boldsymbol{q} + \boldsymbol{u}^{\mathrm{T}}P\boldsymbol{u})\,dt \tag{20}$$

where $S = \operatorname{diag}\{s_{ii}\}$, $Q = \operatorname{diag}\{q_{ii}\}$ and $P = \operatorname{diag}\{p_{ii}\}$, $\Delta\boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}_0$ and $\boldsymbol{q}_0$ is the joint coordinate reference vector.

When acceleration feedforward is used, system (3, 4) becomes

$$\ddot{q} = u + \ddot{q}_0 \tag{21}$$

which can be written

$$\Delta\ddot{q} = u \tag{22}$$

From (22) and (20) the optimal control is found from the n linear quadratic sub-problems

$$\Delta\ddot{q}_i = u_i \tag{23}$$

with performance functional

$$J_i = \tfrac{1}{2}[\Delta q_i(t_f)]^2 s_{ii} + \tfrac{1}{2}\int_{t_0}^{t_f} [(\Delta q_i)^2 q_{ii} + u_i^2 p_{ii}]\, dt \tag{24}$$

which gives the result

$$u_i = g_{i1}\Delta q_i + g_{i2}\Delta\dot{q}_i, \qquad i = 1, \ldots, n \tag{25}$$

where

$$g_{i1} = -p_{ii}^{-1} r_{12}^i \tag{26}$$

$$g_{i2} = -p_{ii}^{-1} r_{22}^i \tag{27}$$

r_{12}^i and r_{22}^i are found from the Riccati equations

$$\dot{r}_{11}^i = p_{11}^{-1}(r_{12}^i)^2 - q_{ii} \tag{28}$$

$$\dot{r}_{12}^i = -r_{11}^i + p_{ii}^{-1} r_{12}^i r_{22}^i \tag{29}$$

$$\dot{r}_{22}^i = -2r_{12}^i + p_{ii}^{-1}(r_{22}^i)^2 \tag{30}$$

with boundary conditions

$$r_{11}^i(t_f) = s_{ii} \tag{31}$$

$$r_{12}^i(t_f) = r_{22}^i(t_f) = 0 \tag{32}$$

The gains g_{i1} and g_{i2} may be computed off-line for the whole trajectory, or they can be determined analytically by assuming $t_f \to \infty$. In this case

$$g_{i1} = -\sqrt{(q_{ii}/p_{ii})}$$

$$g_{i2} = -\sqrt{[2\sqrt{(q_{ii}/p_{ii})}]}$$

However, acceleration feedforward will not give satisfactory results if the reference trajectory has discontinuities in position, velocity or acceleration. Without acceleration feedforward, the model is

$$\ddot{q} = u \tag{33}$$

and the optimal control is then

$$u_1 = g_{i1}x_{1i} + g_{i2}x_{2i} - p_{ii}^{-1}h_2^i \tag{34}$$

h_2^i gives feedforward from future references, and is computed off-line from the adjoint system

$$\dot{h}_1^i = -g_{i1} h_2^i + q_{ii} r_i \tag{35}$$

$$\dot{h}_2^i = -h_1^i - g_{i2} h_2^i \tag{36}$$

$$h_1^i(t_f) = -s_{ii} r_i(t_f) \tag{37}$$

$$h_2^i(t_f) = 0 \tag{38}$$

5. A simulation experiment using the computed torque rechnique

When applying the computed torque technique the decoupling algorithm may be developed in joint space as explained in § 2 or in task space by (Luh, Walker and Paul (1980))

$$\tau = M(q)J^{-1}(q)(u - \dot{J}(q)\dot{q}) - n(q, \dot{q})$$

where $J(q = \partial p/\partial q$ is the manipulator Jacobian, and p is the task space coordinate vector. The state vector of the equivalent model (3, 4) is then

$$x_1 = p \quad \text{and} \quad x_2 = \dot{p}.$$

We turn our attention to the problem of tracking the continuous reference trajectory $r(t)$ in Fig. 1 with a constant valocity. This reference consists of linear and

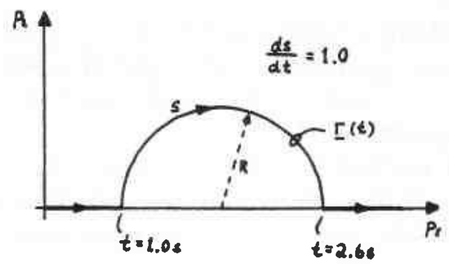

Figure 1. Reference trajectory used in simulation experiment (xy-plot).

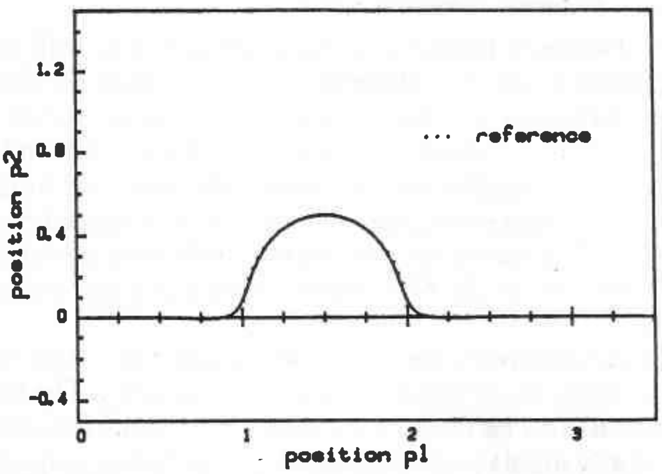

Figure 2. xy-plot of position in simulation experiment.

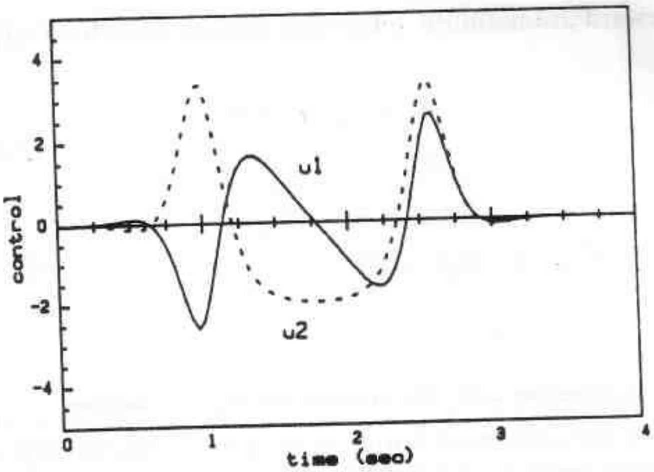

Figure 3. Control variables generating the response in Fig. 2.

nonlinear segments. The splining method is not straightforward in this case because of the nonlinear segment. When optimal control is used as explained above, any reference can be tracked in an optimal manner within the limits set by the dynamics and the workspace of the manipulator.

We used the following controller parameters in the simulation experiment:

$$s_{ii} = 0, \qquad p_{ii} = 1, \qquad q_{11} = 1{\cdot}10^4$$

which gave a closed loop bandwidth of 10 rad/s. The simulation program SIM in the cybernetic program system CYPROS was used.

The parameters describing the reference were $R = 0{\cdot}5$ and $ds/dt = 1{\cdot}0$ (Fig. 1).

A velocity reference was not available. The control u_i was therefore generated by feedback from the states x_{1i} and x_{2i} and feedforward from future values of the position reference $r_i(t)$ as given by (34).

The resulting system tracked the reference accurately except near the corners which were rounded (Fig. 2). The control variables accelerated the system before the change in reference at $t = 1{\cdot}0$ s (Fig. 3) due to the feedforward from future references.

6. Application to a hydraulic industrial robot

The Trallfa TR400 is a hydraulic spray-painting robot, with inertia coupling mainly between joints 2 and 3. Load pressure sensors must be used to apply the computed torque technique on this manipulator. However, servos 2 and 3 were found to be very stiff when independent joint control was used, and it was hard to improve the performance significantly by applying the computed torque technique.

We therefore used independent joint control, and investigated the effect of using linear quadratic tracking theory in combination with these controllers. The robot was controlled with a Motorola 68020/68881 microprocessor with a sampling frequency of 100 Hz.

Only position measurements were available, which meant that we had to use a state estimator to apply linear quadratic tracking rigorously. The reason for this is that state feedback had to be used, as is evident from (13). However, the dynamics associated with the hydraulics were so fast that a sampling frequency of 1000 Hz was required to implement a state estimator. This was not possible as the i/o-card we used had a sampling frequency of 100 Hz. A simple P controller was therefore used at each joint.

Linear quadratic tracking was implemented in two versions.

First, the fast actuator dynamics were neglected in the model (7)–(9) giving a single integration

$$\dot{z}_1 = Kv \tag{39}$$

where v is the control variable. Linear quadratic tracking was then applied rigorously.

In the second implementation, the complete model (7)–(9) was used for feedforward generation.

The performance index

$$J = \frac{1}{2} \int_{t_0}^{\infty} (Qe^2 + Pv^2)\, dt \tag{40}$$

was optimized for the system (39) with $Q = K_p^2$ and $P = 1$. The control deviation is $e = r - z_1$ and r is the position reference to joint i. From (13) the resulting control law is given as:

$$v = -K_p z_1 - Kh_1 \tag{41}$$

where h_1 is found from

$$\dot{h}_1 = KK_p h_1 + K_p^2 r \tag{42}$$

with boundary condition

$$h_1(t_f) = 0 \tag{43}$$

In an experiment on the Trallfa TR400, a saw-tooth on joint 1 was used as a reference. The velocity was $\pm 0{\cdot}5$ rad/s. The bandwidth of servo 1 was $18{\cdot}5$ rad/s.

When only feedback was used,

$$v = K_p(r - z_1) \tag{44}$$

a steady state deviation

$$e = (KK_p)^{-1}\dot{r} \tag{45}$$

was observed (Fig. 4). This is in accordance with the classic servomechanism theory.

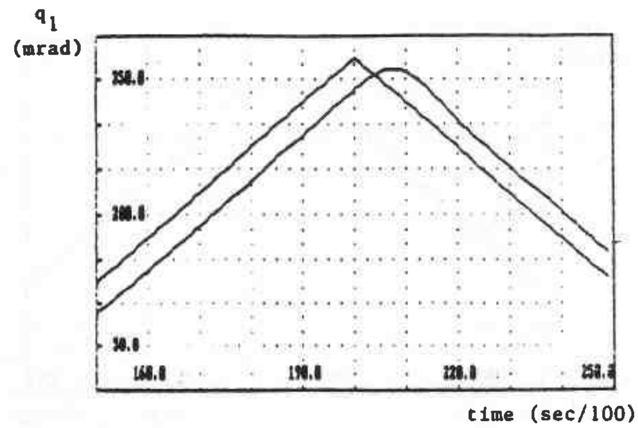

Figure 4. Tracking a saw-tooth reference with joint 1 in TR400 experiment. Proportional feedback was used.

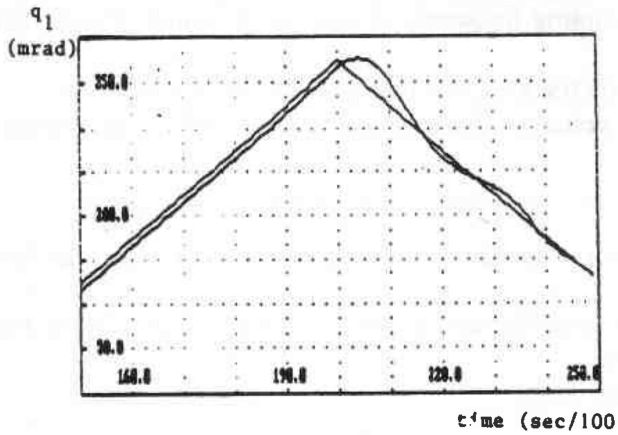

Figure 5. Tracking a saw-tooth reference with joint 1. Proportional feedback and velocity feedforward was used.

The steady-state tracking error can be removed by velocity feedforward:

$$v = K_p(r - z_1) + K^{-1}\dot{r} \tag{46}$$

However, this solution gave an oscillatory result at the corners (Fig. 5).

When linear quadratic tracking was used as given by (41)–(43) the steady-state tracking error was removed, and at the same time, the corner was rounded (Fig. 6). A small steady-state deviation was observed when $\dot{r} > 0$. This was probably due to imperfections in the servovalve.

In the second implementation the feedback control was still

$$u_{fb} = -K_p z_1 \tag{47}$$

but the complete model was used in the computation of the $\boldsymbol{h}$ vector in the control law. Linear quadratic tracking theory cannot be applied directly in this case as the feedback control law is not a result of an optimization of a linear quadratic performance index.

However, a sub-optimal $\boldsymbol{h}$ can be found from the adjoint of the system (7)–(9) with feedback control (47).

In simulations, this method was significantly better at the corner points and for

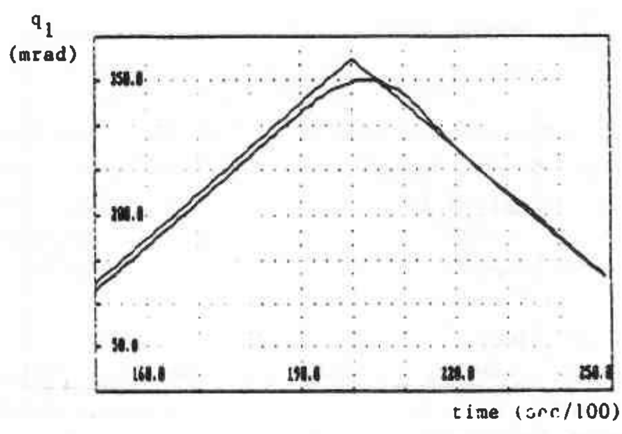

Figure 6. Tracking a saw-tooth reference with joint 1. Proportional feedback and linear quadratic tracking based on a simplified model was used.

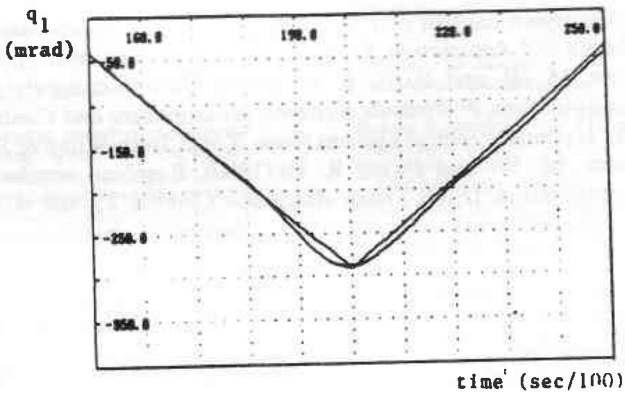

Figure 7. Tracking of saw-tooth reference with joint 1 in TR400 experiment. Proportional feedback and linear quadratic tracking was used.

general trajectories, while it gave the same results as the simpler method for steady-state tracking of a ramp. However, in an experiment, the results were similar to those obtained with the simplified model (Fig. 7).

7. Conclusion

It has been demonstrated that linear quadratic optimal tracking gives very satisfactory results in manipulator control. When the reference trajectory consists of straight-line segments, the method produces almost the same result as splining and ideal feedforward. The method may also be used for complex reference trajectories where splining methods are hard to apply.

In a simulation experiment, this method was used in combination with the computed torque technique. The system tracked a reference with straight-line segments connected to a circular arc. The transitions between the straight line and the circular arc were smooth and accurate.

The method has also been applied to the hydraulic Trallfa TR400 spray-painting robot. A saw-tooth reference was tracked by joint 1. The steady-state error was eliminated when the linear quadratic tracking system was used, and the corner was rounded extremely well.

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References

Paul, R. P. (1981). *Robot manipulators: Mathematics, programming and control*, (Cambridge, Massachusetts: The MIT Press).

Craig, J. J. (1986). *Introduction to robotics* (Reading, Massachusetts: Addison-Wesley).

Athans, M. and Falb, P. L. (1966). *Optimal Control* (New York: McGraw-Hill).

Bejczy, A. K. (1974). Robot arm dynamics and control, Jet Propulsion Lab., Pasadena, CA, Tech. Memo 33–669.

Egeland, O. (1987). Cartesian control of a hydraulic redundant manipulator, in *Proc. IEEE Int. Conf. Robotics and Automation.* Raleigh, North Carolina, Mar., pp. 2081–2086.

Luh, J. Y. S., Walker, M. W. and Paul, R. P. (1980). On-line computational scheme for mechanical manipulators, *J. Dynamic Systems, Measurement and Control*, **102,** 69–76.

Merritt, H. E. (1976). *Hydraulic control systems* (New York: John Wiley & Sons).

Luh, J. Y. S., Walker, M. W. and Paul, R. P. (1980). Resolved acceleration control of mechanical manipulators, *IEEE Trans. Automatic Control*, **25,** 468–474.