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Stability of Pareto-optimal allocations of resources to activities

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A concept of stability is introduced for the Pareto-optimal solutions of a vector-valued problem of the allocation of resources to activities, and characterized by a property which is independent of uncertainties in the efficiency matrix of the allocations. Any feasible solution can be improved by cyclic shifts to give a stable Pareto-optimal solution. The resource allocation problem of the maximization of the sum of the utility returns from the activities and a problem with fuzzy resources and activities are shown to have stable Pareto-optimal solutions.

1. Introduction

A multi-objective resource allocation problem, denoted by M, is defined as follows:

$$M\text{: Vector-maximize } z(x) = (x_1, \ldots, x_k, \ldots, x_K) \tag{1}$$

where

$$x = (x_{jk}) \text{ is the } J \times K \text{ allocation matrix} \tag{2}$$

and

$$x_k = \sum_{j=1}^{J} \alpha_{jk} x_{jk} \quad \text{for} \quad k = 1, \ldots, K \tag{3}$$

subject to the constraints:

$$\sum_{k=1}^{K} x_{jk} = h_j; \quad j = 1, \ldots, J \tag{4}$$

and

$$x \geqslant 0 \tag{5}$$

J is the number of resources, K the number of activities, x_{jk} the amount of resource j allocated to activity k, h_j the available quantity of resource j, and α_{jk} the effectiveness of resource j when allocated to activity k. Numerical values satisfy: $J \geqslant 1$, $K \geqslant 2$, $h_j > 0$ for all j; $\alpha_{jk} \geqslant 0$ for all (j, k) and for any j there exists a $k(j)$ such that $\alpha_{jk(j)} > 0$.

The matrix $\alpha = (\alpha_{jk})$ is called the efficiency matrix.

A feasible solution x of P is Pareto-optimal if and only if there is no feasible solution x' such that

$$z(x') \geqslant z(x)$$

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where '$\geqslant$' denotes '$\geqq$ but not $=$' with $\geqq$ denoting weak componentwise inequality.

Clearly there is no restriction in generality by assuming that $x_{jk} > 0$ only if $\alpha_{jk} > 0$.

For a given feasible solution $x = (x_{jk})$ of the problem M, Danskin (1967) defined a cycle of allocations to be a sequence given by:

$$x_{j_1k_1}\, x_{j_1k_2}, \ldots, x_{j_ik_i}\, x_{j_ik_{i+1}}, \ldots, x_{j_{n-1}k_{n-1}}\, x_{j_{n-1}k_n} \tag{6}$$

where $k_n = k_1$, and Einbu (1978) defined the associated α-cycle by:

$$\alpha_{j_1k_1}\, \alpha_{j_1k_2}, \ldots, \alpha_{j_ik_i}\, \alpha_{j_ik_{i+1}}, \ldots, \alpha_{j_{n-1}k_{n-1}}\, \alpha_{j_{n-1}k_n} \tag{7}$$

and the ratio ρ of the cycles (6, 7) by:

$$\rho = \prod_{i=1}^{n-1} (\alpha_{j_ik_i}/\alpha_{j_ik_{i+1}}) \tag{8}$$

An activity of the set $\{k_1, k_2, \ldots, k_n\}$ is called an activity of the cycle (6).

Mjelde (1983) demonstrated the following result:

Theorem (1)

A feasible solution x of M is Pareto-optimal if and only if one of the following two conditions is satisfied:

(1) There is no cycle (6) of positive allocations, or otherwise
(2) The ratio of all associated α-cycles (7) is equal to 1.

The purpose of this paper is to introduce the concept of stable Pareto-optimal solutions of the problem M, and to describe the set of all such solutions by a simple characterization which is independent of properties of the efficiency matrix α, and, in particular, independent of the existence of α-cycles with ratio 1. It follows that stable Pareto-optimal solutions contain at most $(J + K - 1)$ non-zero allocations. It is then shown how any feasible solution $x = (x_{jk})$ of M, not satisfying the property of stable Pareto-optimality, can be used to derive associated feasible solutions of M that satisfy this property, and without a decrease of the corresponding values of x_k given by eqn. (3). It is observed that there exists a stable Pareto-optimal solution which is optimal for the resource allocation problem of the maximization of the utility returns from several activities. This is also the case for the allocation problem with fuzzy resources and fuzzy activities discussed by Mjelde (1986), when the h_j in eqn. (4) are replaced by the actual resource consumptions in the fuzzy problem.

2. Stable Pareto-optimality

Definition (1)

A Pareto-optimal solution x of the problem M is said to be stable if and only if there exists a real number $\Delta > 0$ such that x is still Pareto-optimal if (α_{jk}) in eqn. (3) is replaced by any (α'_{jk}) satisfying the following requirement:

$$\alpha'_{jk} \in [\alpha_{jk} - \Delta, \alpha_{jk} + \Delta] \quad \text{for} \quad j = 1, \ldots, J;\ k = 1, \ldots, K \tag{9}$$

Such a solution x is also said to be a stable Pareto-optimal solution.

In a practical sense only Pareto-optimal solutions that are stable are of interest, since there are always uncertainties associated with the efficiency-data α_{jk}.

A stable-Pareto-optimal solution of M cannot contain a cycle (6) with ratio $\rho = 1$, since if $\rho = 1$ the definition of (α'_{jk}) given by

$$\alpha'_{j_i k_i} = \alpha_{j_i k_i} + \Delta; \quad i = 1, \ldots, (n-1) \tag{10}$$

$$\alpha'_{j_i k_{i+1}} = \alpha_{j_i k_{i+1}}; \quad i = 1, \ldots, (n-1) \tag{11}$$

for a $\Delta > 0$, demands that

$$\rho' = \prod_{i=1}^{n-1} (\alpha'_{j_i k_i}/\alpha'_{j_i k_{i+1}}) > 1 \tag{12}$$

By theorem (1) this demands that the cycle (6) is not Pareto-optimal when (α_{jk}) is replaced by (α'_{jk}). The implication is that a stable Pareto-optimal solution of M contains no cycle (6) of positive allocations; conversely, such a solution is still Pareto-optimal if (α_{jk}) is replaced by an (α'_{jk}) satisfying the requirement (q), because x remains unchanged by this replacement. The following result has been demonstrated:

Theorem (2)

x is a stable Pareto-optimal solution of M if and only if there is no cycle (6) of positive allocations.

The observation that (x_{jk}) must contain a cycle if it contains at least $(J + K)$ positive allocations x_{jk} gives:

Theorem (3)

Stable Pareto-optimal solutions of M contain at most $(J + K - 1)$ non-zero allocations.

Einbu (1981) improved solutions of a return-maximization problem by a procedure based on cyclic shifts of the allocations; it will be demonstrated that a similar procedure is applicable to the vector-maximization problem M.

A shift along a cycle (6) of positive allocations is defined by the replacement of $(x_{jk}) \geqslant 0$ by $(x'_{jk}) \geqslant 0$ where:

$$x'_{j_i k_i} = x_{j_i k_i} - \Delta_i; \quad i = 1, \ldots, (n-1) \tag{13}$$

$$x'_{j_i k_{i+1}} = x_{j_i k_{i+1}} + \Delta_i; \quad i = 1, \ldots, (n-1) \tag{14}$$

$$\Delta_i \alpha_{j_i k_{i+1}} - \Delta_{i+1} \alpha_{j_{i+1} k_{i+1}} = 0; \quad i = 1, \ldots, (n-2) \tag{15}$$

$$x'_{jk} = x_{jk} \quad \text{if} \quad (j, k) \notin \{(j_i, k_i), (j_i, k_{i+1})\} \quad \text{for} \quad i = 1, \ldots, (n-1) \tag{16}$$

for selected values of Δ_i for $i = 1, \ldots, (n-1)$.

The eqns. (13, 14) imply that (x'_{jk}) is a feasible solution of M with x'_k of equation (3) satisfying:

$$x'_k = x_k \quad \text{for} \quad k \neq k_1 \tag{17}$$

$$x'_{k_1} - x_{k_1} = \Delta_1 \alpha_{j_1 k_1}[(1/\rho) - 1] \tag{18}$$

demanding that $x'_{k_1} = x_{k_1}$ if $\rho = 1$ and $x'_{k_1} > x_{k_1}$ if Δ_1 is chosen such that $\Delta_1 < 0$ if $\rho > 1$ and $\Delta_1 > 0$ if $\rho < 1$. Furthermore, Δ_1 can be chosen such that the cycle (6) is removed by making at least one of its allocations equal to zero, as follows:

Define:

$$\xi_{i-1} = \prod_{q=1}^{i-1} (\alpha_{j_{q+1}k_{q+1}}/\alpha_{j_q k_{q+1}}) \tag{19}$$

Then:

$$x'_{j_m k_m} = 0 \quad \text{if} \quad \rho < 1 \quad \text{and} \quad \Delta_1 = x_{j_m k_m}\xi_{m-1} = \text{Min}\,[x_{j_i k_i}\xi_{i-1}]$$
$$\text{for} \quad m \in \{1, \ldots, (n-1)\} \quad i = 1, \ldots, (n-1) \tag{20}$$

$$x'_{j_m k_{m+1}} = 0 \quad \text{if} \quad \rho \geqslant 1 \quad \text{and} \quad \Delta_1 = -x_{j_m k_{m+1}}\xi_{m-1} = \text{Max}\,[-x_{j_i k_{i+1}}\xi_{i-1}]$$
$$\text{for} \quad m \in \{1, \ldots, (n-1)\} \quad i = 1, \ldots, (n-1) \tag{21}$$

Given a feasible solution x of M, the process defined above can be repeated until all cycles (6) of x has been removed. According to theorem (2) this produces a stable Pareto-optimal solution of M, and the following result has been demonstrated:

Theorem (4)

Any feasible solution x of M can be improved to give a stable Pareto-optimal solution x' of M such that $x'_k \geqslant x_k$ for $k = 1, \ldots, K$, by a finite number of shifts along cycles (6) defined by the eqns. (13–16) and (19–21). For a cycle (6) with $\rho \neq 1$ there is shift which makes $x'_k > x_k$ for any selected activity k of the cycle.

3. Return maximization and fuzzy resource allocation

Consider the problem, denoted by R, of the maximization of the total return from several activities as follows:

$$R\colon \text{maximize } z = \sum_{k=1}^{K} r_k(x_k)$$

subject to the constraints (2, 3, 4, 5) of M, where r_k are non-decreasing and concave functions with $r_k(0) = 0$ describing the results from the activities k.

Since Einbu (1981) demonstrated that cycles may be removed in an optimal solution of R by the previously given transformations (13–16) and (19–20) satisfying the properties (17, 18), theorem (2) gives the result of:

Theorem (5)

There is a stable Pareto-optimal solution of M which is optimal for the return maximization problem R.

Mjelde (1986) derived results applicable to the following problem F of the allocation of fuzzy resources to fuzzy activities:

$$F\colon \text{Maximize } z = \text{Min}\left\{\min_{j=1,\ldots,J} \mu_j\left(\sum_{k=1}^{K} x_{jk}\right),\ \min_{k=1,\ldots,K} r_k\left(\sum_{j=1}^{J} \alpha_{jk}x_{jk}\right)\right\}$$

subject to the single constraint $(x_{jk}) \geqslant 0$.

In this formulation each μ_j is a strictly decreasing function with $\mu_j(0) > 0$ and $\mu_j(y_j) = 0$ if $y_j > H_j$ for $H_j > 0$; and each r_k is a strictly increasing function with $r_k(0) \geqslant 0$.

Since, for given values of

$$y_j = \sum_{k=1}^{K} x_{jk}$$

for $j = 1, \ldots, J$, the previously given transformations (13–16) and (19–20) may be used to remove any cycles of a feasible solution of F, keeping the values of y_j unchanged for $j = 1, \ldots, J$ and without a decrease of any of the x_k given by eqn. (3), it follows that the following theorem is valid:

Theorem (*6*)

There is an optimal solution of the fuzzy resource allocation problem F, which is also a stable Pareto-optimal solution of the problem M with h_j in the constraint (4) replaced by the values of y_j corresponding to an optimal solution of F.

References

Danskin, F. M., (1967), *The theory of max–min* (Berlin, Springer-Verlag).

Einbu, J. M., (1978), Optimal allocation of continuous resources to several activities with a concave return function — some theoretical results'. *Math. Ophs. Res.*, **3**, 82–88.

Einbu, J. M., (1981), Improving solutions of a class of allocation problems by cyclic shifts of resources. *Opl. Res. Q.*, **32**, 401–404.

Mjelde, K. M., (1983), Properties of Pareto-optimal allocations of resources to activities. *Modeling, Identification and Control*, **4**, 167–173.

Mjelde, K. M., (1986), Fuzzy resource allocation. *Fuzzy Sets and Systems*, **19**, 239–250.