MODELING, IDENTIFICATION AND CONTROL, 1986, VOL. 7, NO. 2, 83–92
doi:10.4173/mic.1986.2.3

Diversifying the risk associated with exploration

SJUR D. FLÅM† and SVERRE STORØY‡

Keywords: *stochastic models, entropic penalty, portfolio problems.*

This paper is concerned with the allocation of exploratory efforts under the limitation of a fixed budget. A chance constrained problem is formulated. To solve this problem an algorithm is developed which is based on the entropic penalty approach recently presented by Ben-Tal.

1 Introduction

This paper is concerned with the allocation of exploratory effort under the limitation of a fixed budget. The budget constraint could reflect attitudes of the capital markets that providing additional funds for exploration would increase the risk of bankruptcy so much that a feasible interest rate would not give adequate compensation. By contrast the expenditure limit on exploration may be self-imposed in order to preserve corporate control or to avoid increasing the organization's size so fast that inefficiences would result.

In any case, managers are faced with a *portfolio problem.* How should scarce capital resources be allocated to explore various risky prospects? The standard portfolio problem in the financial literature takes on the following form:

$$\text{(PS) maximize } Eu(w_0 \sum_{i=1}^{n} \xi_i x_i)$$

subject to

$$\sum_{i=1}^{n} x_i = 1, \quad x_1, \ldots x_n \geq 0$$

Here x_i is the proportion of the budget w_0 invested in prospect i. The return distribution $\xi = (\xi_1, \ldots \xi_n)$ is most often assumed to be multinormal with known mean vector and covariance matrix. As formulated here the decision vector x and the return ξ have the same dimension. This does not exclude, of course, that ξ is generated by fewer genuinely independent variables. A salient feature of (PS) is the concavity and the monotonicity of the utility function u. These two properties reflect the risk aversion and the desire for end-of-period wealth. Finally the non-negativity constraints prohibit short sales. For an analysis of (PS) and references see Kallberg and Ziemba (1984), Pulley (1983). This paper is prompted partly by two objections to the problem formulation (PS).

First, what is the right choice of u? Second, the observed data on economic returns often refute the assumption about normality. Notably this is so in the important case of petroleum exploration.

Received 15 November 1985.

† Chr. Michelsen Institute, Dept. of Science and Technology, N–5035 Fantoft, Bergen, Norway.

‡ University of Bergen, Dept. of Informatics, Allégt. 55, N–5000 Bergen, Norway.

We have responded to these objections in the following manner. First, risk aversion is excluded from the criterion and rather taken care of by means of a chance constraint. Specifically, we shall maximize expected revenues in place of expected utility. Yet we acknowledge the preference for safety by requiring that the portfolio should furnish a minimal profit with a certain probability p. The exact figure p now accounts for the attitude towards risk, with $p = 1$ meaning full emphasis on secure returns, and $p = 0$ reflecting the preference of an active risk seeker. We shall find it particularly interesting to explore how the optimal portfolio depends on p.

Secondly, we shall adapt the distributional assumptions of the portfolio problem more to the reality of petroleum exploration by allowing for returns to follow any statistical distribution which has a density. As already mentioned the normal density does not describe reality very well. Several other distributions have been used to model oil and gas field-size; among the more frequent ones are the log-normal, dating back to Allais (1957), and the gamma (including the exponential). In this paper we shall not specify the class to which the probability distribution belong. The alternative problem formulation which is to replace (PS) and the associated algorithm will both tolerate rather general continuous distributions. Only when we come to computations shall we resort to the convenient and realistic log-normal specification.

2 The model

This section introduces a formal version of the decision problem. In order to fix ideas we shall henceforth speek about petroleum exploration. This will, however, not limit the range of possible applications. As already mentioned we choose to replace the portfolio selection problem (PS) with a *chance constrained problem* (CC). The objective is to maximize expected revenues subject to a probabilistic constraint on the profitability of the portfolio. In formal terms we seek to

$$\text{(CC) maximize } \bar{\pi}x = \sum_{i=1}^{n} \bar{\pi}_i x_i$$

subject to

$$P_{f_0}[\pi(R)x \geq \pi \min] \geq p, \tag{1}$$

$$Ax = b, \tag{2}$$

and

$$x \min \leq x \leq x \max,\ x,\ \pi(R),\ \bar{\pi} \in R^n : x \text{ column, } \pi(R) \text{ and } \bar{\pi} \text{ rows.} \tag{3}$$

Here, for $i = 1, \ldots, n$,

x_i = the proportion of prospect i bought by the company, this proportion being directly constrained by (3), that is

$$0 \leq x \min_i \leq x_i \leq x \max_i \leq 1, \text{ for every } i;$$

$\pi_i(R)$ = after tax present value of prospect i contingent on the recoverable reserves R being revealed there. In expectation this present value equals

$\bar{\pi}_i$ = $E\pi_i(R)$;

$\pi \min$ = minimal level of desired profit;

p = lower bound on the probability of achieving at least the minimal profit π min,

P_{f_0} = the probability distribution induced by the density f_0.

The constraint $Ax = b$ of (CC) comprise at least one important restriction, namely the limit on expenses imposed by the fixed budget. Specifically, the first row of $Ax = b$ takes on the form

$$cx = w_0, \; c \in R^n \text{ (row)},$$

where

c_i = the cost of exploring prospect i alone, and
w_0 = the total budget for exploration.

In addition to $cx = w_0$, several other constraints may be in force to account for ties between different prospects or to reflect concerns about the geographical distribution of the portfolio. We shall not elaborate more on these additional constraints here. Their precise nature will certainly depend on the practical case at hand. The constraint $cx = w_0$, will never be neglected, however. The joint probability distribution of $R = (R_1, \dots, R_k)$ is assumed to be known and to have a joint density $f_0(R)$. Note that the dimension of R does not need to coincide with that of x. In fact, the return on different prospect may in reality be determined by a small set of common factors. Also note that so far we have imposed no assumptions on the probability distribution apart from the existence of the density f_0. The notation P_{f_0} in (CC) is meant to convey that the right tail probability has been calculated by means of the known density f_0. We acknowledge that the precise identification of f_0 is a demanding task, see Barouch, Kaufman (1977), Schuenemeyer, Drew (1983). Here the distribution is simply taken as data.

3 Solution procedure

In this section we outline a procedure for actually solving the chance constrained program (CC). In practice we shall have to contend ourselves with approximate solutions. The main approach is designed by Ben-Tal (1985), and we call it *the entropic penalty method.*

The main strategy is to relax the troublesome profitability constraint (1) of (CC) by penalizing solutions that violate it. The precise nature of the penalty function will soon be given. First it is appropriate to review highlights of the computational method of exterior penalization as it applies to our problem.

In place of solving (CC) we shall consider the *relaxed chance constrained* problem

$$(\mathrm{RCC}_\theta) \; \min P(x, \theta) = -\bar{\pi}x + \theta P_E(x)$$

over all $x \in R^n$ such that (2) and (3) are satisfied.

Here $\theta > 0$ plays the role of a parameter and P_E is a *penalty function* to be defined. It has the following property:

$$P_E(x) = \begin{cases} 0 \text{ if } p(x) = P_{f_0}[\pi(R)x \geq \pi \min] \geq p \\ > 0 \text{ otherwise} \end{cases}$$

We note that the profitability constraint (1) of (CC) is now removed and is included in the criterion by means of the term $\theta P_E(x)$. Thus (RCC_θ) is linearly constrained and therefore the remaining constraints are relatively easy to handle.

In principle the penalty method (see e.g. Luenberger (1984)) proceeds by increasing θ until we are satisfied with the resulting optimal solution $x(\theta)$. The justification for this approach lies partly in the fact that with

$$\theta_2 > \theta_1,$$

the following inequalities hold:

$$P_E(x(\theta_2)) \leq P_E(x(\theta_1)). \tag{4}$$

$$\bar{\pi}x(\theta_2) \leq \bar{\pi}x(\theta_1), \tag{5}$$

$$P(x(\theta_1),\theta_1) \leq P(x(\theta_2), \theta_2)$$

(4) and (5) tell that as θ is increased, $x(\theta)$ becomes 'more feasible' at the expense of lowering the value of the criterion. In the limit as $\theta \to +\infty$, we obtain

$$\lim_{\theta \to +\infty} \theta P_E(x(\theta)) = 0 \tag{6}$$

and

$$\lim_{\theta \to +\infty} \bar{\pi}x(\theta) \geq \sup \text{(CC)} \tag{7}$$

(6) and (7) suggest that asymptotically $x(\theta)$ becomes feasible and solves the problem. In fact, if $\theta_k \to +\infty$ and x is an accumulation point of

$$\{x(\theta_k)\}_{k=1}^{\infty},$$

then $P_E(x) = 0$ and

$$\bar{\pi}x = \sup \text{(CC)}.$$

For a proof of these results we refer to Mangasarian (1984).

We also want to emphasize that in order to obtain an approximate solution within a desired feasibility tolerance $\delta > 0$ we need only solve (RCC_θ) for two values of θ. Namely, let x be feasible for (CC) and $x(\theta_1)$ an optimal solution of (RCC_{θ_1}).

If

$$\bar{\pi}\hat{x} \geq \bar{\pi}x(\theta_1),$$

then $\hat{x}$ is optimal, else for

$$\theta_2 \geq \theta_1,\ \theta_2 \geq \frac{\bar{\pi}x(\theta)_1 - \bar{\pi}\hat{x}}{\delta}$$

it follows that

$$P_E(x(\theta_2)) \leq \delta$$

(see Mangasarian (1984)). It remains to provide a precise definition of the penalty function $P_E(x)$. It is given in terms of the relative entropy functional, and is accordingly called *entropic penalty*. It is defined as:

$$P_E(x) = \inf I(f, f_0)$$

where

$$I(f, f_0) = \int f(R) \log \frac{f(R)}{f_0(R)}\, dR$$

is the entropic penalty, and where the infimum is taken over all probability densities f such that $P_f(\pi(R)x \geq \pi \text{ min}) \geq p$.

A few words about the intuition behind this approach is warranted. For a given allocation x it may happen that the profitability constraint

$$p(x) = P_{f_0}[\pi(R)x \geq \pi \text{ min}] \geq p$$

is satisfied.

If so, we incur no penalty. By contrast, if this constraint is violated, we face a certain additional cost. This cost is proportional to the 'distance' $I(f, f_0)$ from f_0 to the closest density f for which the profitability constraint is again satisfied.

So far the complexity of the formula for P_E gives the impression that nothing is gained. However a *dual representation* of P_E is available which enables us to compute P_E rather easily. For the general chance constrained program

$$\inf \{g_0(x) : Eg(x,R) \geq a\}$$

the dual representation of the entropic penalty is given by

$$P_E(x) = \sup_{y \geq 0} \{ya - \log \int f_0(R) e^{yg(x,R)} dR\},$$

$$g,\ a,\ y \in R^I;\ y \text{ row and } g,\ a \text{ columns},$$

see Ben-Tal (1985).
Now $I = 1$ in the following.
In our case,

$$g_0(x) = \begin{cases} -\bar{\pi}x \text{ if } Ax = b,\ x \text{ min} \leq x \leq x \text{ max} \\ +\infty \text{ otherwise,} \end{cases}$$

$a = p$, and

$$g(x, R) = \begin{cases} 1 \textit{ if } \pi(R)x \geq \pi \text{ min} \\ 0 \text{ otherwise} \end{cases}$$

Recall that

$$p(x) = P_{f_0}[\pi(R)x \geq \pi \text{ min}].$$

Now introduce

$$q(x) = 1 - p(x) \text{ and } q = 1 - p.$$

Then

$$P_E(x) = \sup_{y \geq 0} \{py - \log[e^y p(x) + q(x)]\}$$

$$= \begin{cases} 0 \text{ if } p(x) \geq p, \text{ and} \\ p \log \left(\frac{p}{q} \Big/ \frac{p(x)}{q(x)}\right) - \log \left\{q(x)\left(1 + \frac{p}{q}\right)\right\} \text{ otherwise.} \end{cases}$$

The upshot of all this is that instead of solving (CC) we shall prefer to solve

$$(\text{RCC}_\theta) \min -\bar{\pi}x + \theta\{py - \log[e^y p(x) + q(x)]\}$$

s.t.

$$y = \max\left(0, \log\left(\frac{p}{q} \Big/ \frac{p(x)}{q(x)}\right)\right).$$

$$Ax = b,$$

and

$$x \text{ min} \leq x \leq x \text{ max}$$

for appropriate values of θ.

To actually solve practical instances of (RCC_θ) involving several linear constraints we suggest the reduced gradient method, or variants of it, see Luenberger (1984). Here we proceed to study the simple case with only one linear constraint, namely the limitation

$$cx = w_0$$

imposed by the budget. Already at this level it will be seen that computation is fraught with some difficulties. One reason is that usually no closed expression obtains for

$$p(x) = P_{f_0}[\pi(R)x \geq \pi \text{ min}]$$

or its gradient. Therefore one must contend with approximate values produced by Monte Carlo simulation. The next section will exemplify this.

Another cause of troubles is that the criterion of (RCC_θ) need not be convex. Thus we face the possibility of producing only local optima in each iteration of the method. For a discussion of this see Flåm, Pinter (1985).

4. The case with only one linear constraint imposed by the budget

This section sets the solution procedure of section 3 to work on the case where $Ax = b$ amounts to the single budget constraint

$$cx = w_0$$

and the reserves

$$R = (R_1, \ldots, R_k)$$

are multi-log normally distributed.

The parameters of this distribution are known, specifically ($\log R_1, \ldots, \log R_k$) is multi-normally distributed with specified mean vector $\mu = (\mu_i) = (E \log R_i)$ and covariance matrix

$$\sum = (\sigma_{ij}) = (E(\log R_i - \mu_i)(\log R_j - \mu_j))$$

$i, j = 1, \ldots k$, see Ferguson (1967).

We shall invoke the *nondegeneracy assumption* that $c_i \neq 0$ for all i. According to the classical Cholesky-factorisation (see e.g. Golub and van Loan (1983)), there exists a unique lower triangular matrix

$$C = \begin{bmatrix} c_{11} & & 0 \\ c_{21} & c_{22} & \\ c_{k1} & c_{k2} & c_{kk} \end{bmatrix}$$

such that

$$\sum = CC^T.$$

In fact, it is given, recursively for

$$i = 1, \ldots, k, \quad j = 1, \ldots, i$$

by

$$c_{ij} = \frac{\sigma_{ij} - \sum_{s=1}^{j-i} c_{is}c_{js}}{\left(\sigma_{jj} - \sum_{s=1}^{j-i} c_{js}^2\right)^{1/2}}$$

where

$$\sum_{k=1}^{0} = 0$$

Now the procedure for generating a lognormal $(R_1, \ldots, R_k)$ is as follows:

1. If k is even, generate k independent variables $U_1, \ldots, U_k$ from the standard uniform distribution over the interval [0, 1]. (This might be done by a random number generator). If k is odd, generate an additional independent standard uniform variate U_{k+1}.

2. Let

$$X_1 = (-2 \ln U_1)^{1/2} \cos 2\pi U_2.$$
$$X_2 = (-2 \ln U_1)^{1/2} \sin 2\pi U_2.$$
$$X_3 = (-2 \ln U_3)^{1/2} \cos 2\pi U_4,$$
$$X_4 = (-2 \text{ in } U_3)^{1/2} \sin 2\pi U_4$$

and so on. In general,

$$X_{2s-1} = (-2 \ln U_{2s-1})^{1/2} \cos 2\pi U_{2s}$$
$$X_{2s} = (-2 \ln U_{2s-1})^{1/2} \sin 2\pi U_{2s},$$

$$\text{for } s = 1, \ldots, m \text{ where } m = \begin{cases} \frac{k}{2} \text{ if } k \text{ is even} \\ \frac{k+1}{2} \text{ if } k \text{ is odd} \end{cases}$$

Then $X_1, \ldots, X_k$ are independent standard normal variables. This approach is due to Box and Muller, see Rubinstein (1981) and the references therein.

3. Let $Y = XC^T + \mu$,

where C is 'the square root' of $\sum$ and

$$\mu = (\mu_1, \ldots, \mu_k).$$

Then Y is multinormally distributed with mean vector μ and covariance matrix $\sum$.

4. Let $R = (e^{Y_1}, \ldots, e^{Y_k})$

Now being furnished with the ability to generate lognormal variates we turn to the simple version of

$$(\text{RCC}_\theta) \min f(x) := -\bar{\pi}x + \theta\{py - \log [e^y p(x) + q(x)]\}$$

s.t.

$$y = \max\left(0, \log\left(\frac{p}{q}\Big/\frac{p(x)}{q(x)}\right)\right)$$

$$cx = w_0$$

$$x \min \leq x \leq x \max$$

To solve (RCC_θ) we shall need to produce surrogate values for f and its gradient

$$\nabla f(x) = -\pi - \theta \frac{e^y \nabla p(x) + \nabla q(x)}{e^y p(x) + q(x)}$$

by estimating, $p(x)$ and $\nabla p(x)$.
The estimate for $p(x)$ (and $q(x)$) we obtain by Monte Carlo simulation. At the same time we estimate $\nabla p(x)$(and $\nabla q(x)$) as finite differences. The estimation of $p(x)$ may be done as follows:
Let Q be a sufficient large positive integer. Then generate Q versions of R (by the procedure described):

$$R^{(1)}, \ldots, R^{(Q)}$$

Then for a given x we estimate $p(x)$ as:

$$p(x) := \frac{1}{Q} (\# R^{(i)} x \geq \pi \min)$$

The algorithm below for solving (RCC_θ) is an adaption of the reduced gradient method (see e.g. Luenberger (1984)). The notation used is the following:
Let x be such that

$$cx = w_0 \text{ and } x \min \leq x \leq x \max \tag{8}$$

Pick any non-zero (basic) coordinate x_B of x and call it the *dependent* variable. Denote the remaining (nonbasic) *independent* part of x by x_N.
Then

$$x_B = c_B^{-1} w_o - c_B^{-1} c_N x_N, \tag{9}$$

and the gradient of $f(x)$ with respect to x_N is

$$r = -\bar{\pi}_N - \theta \frac{e^y \nabla_N p(x) + \nabla_N q(x)}{e^y p(x) + q(x)} + \bar{\pi}_B c_B^{-1} c_N$$
$$+ \theta \left(\frac{e^y \nabla_B p(x) + \nabla_B q(x)}{e^y p(x) + q(x)} \right) c_B^{-1} c_N \tag{10}$$

Now the algorithm is as follows:

0. Initialization : Given/find x such that (8) is satisfied.
1. Compute directions : Find (9) and an estimate of (10) by using estimates of $p(x)$, $\nabla p(x)$, $q(x)$ and $\nabla q(x)$.

   Set

$$(\Delta x_N)_i := \begin{cases} -r_i \text{ if } x \min_i < x_i < x \max_i \\ \text{or } (x_i = x \min_i \text{ and } r_i < 0) \\ \text{or } (x_i = x \max_i \text{ and } r_i > 0 \\ 0 \text{ otherwise} \end{cases}$$

2. Use directions : If $\Delta x_N = 0$ stop, x is a solution to (RCC_θ)

$$\text{else let } \Delta x_B = -c_B^{-1} c_N \Delta x_N,\ \Delta x = (\Delta x_B, \Delta x_N)$$

   Find a minimal α min and a maximal α max such that

$$\text{for all } \alpha \in [\alpha \min, \alpha \max] \text{ we have } x \min \leq x + \alpha \Delta x \leq x \max$$

3. Line search : minimize $f(x + \alpha\Delta x)$ over $\alpha \in [\alpha \min, \alpha \max]$,

let $\bar{\alpha}$ be the solution

set

$$x := x + \bar{\alpha}\Delta x$$

4. Test : If $\alpha \min < \bar{\alpha} < \alpha \max$ then go to 1, else declare the vanishing variable independent and declare a strictly positive variable in the independent set dependent. Go to 1.

The computational experience with this algorithm so far is rather limited. However, some problems with both artificial and real life data have been solved successfully. Some notes are to be made:

The convergence of the algorithm seems to be rather slow ($O(n^3)$ to $O(n^4)$ iterations).

If θ is selected to be great (i.e. we want to come close to a solution to (CC) at once), the classical jamming property with this type of method becomes apparent (see Luenberger (1984)).

The need for using great values of θ is rather limited since we only need to apply the (RCC_θ) algorithm with two different values of θ in order to get a solution within a given feasibility tolerance, provided this tolerance is not set too restrictive.

Neither the convergence nor the solution itself seems to be sensitive on variations in the sample size Q. In our experiments so far we have tried different values of Q varying from 10 to 10^3.

Acknowledgement

The paper has benefited substantially from comments and suggestions given by an anonymous referee.

REFERENCES

ALLAIS, M. (1957). Method of Appraising Economic Prospects of Mining Exploration over Large Territories, *Management Science*, **3**, 285–347.

BAROUCH, E., and HAUFMAN, G. M. (1977). Estimation of undiscovered Oil and Gas, *Proceedings of the Symposium in Applied Math.*, American Mathematical Society, **21**, 77–91.

BEN–TAL, A. (1985). The Entropic Penalty Approach to Stochastic Programming, *Mathematics of Operations Research*, **10**, 2.

FERGUSON, R. (1967). *Mathematical Statistics, A Decision Theoretic Approach* (Academic Press, New York).

FLÅM, S. D., and PINTER, J. (1985). Selecting Oil Exploration Strategies: Some Stochastic Problem Formulations and Solution Methods, CMI-Report no. 852611-1, Chr. Michelsen Institute, Bergen.

GOLUB, G. H., and VAN LOAN, C. F. (1983). *Matrix Computations* (John Hopkins University Press, Baltimore).

KALLBERG, F. G., and ZIEMBA, W. T. (1984). Mis-Specifications of Portfolio Selection Problems, pp. 74–87 in G. Bamberg and K, Spreman (eds): *Risk and Capital*, Lecture Notes in Economics and Mathem. Systems, 227, Springer Verlag, N. Y.

LUENBERGER, D. G. (1984). *Linear and Nonlinear Programming*, Second Edition, Addison Wesley.

MANGASARIAN, O. L. (1984). Some Application of Penalty Functions in Mathematical Programming, Computer Sciences Technical Report # 544, University of Wisconsin.

Pulley, L. B. (1983). Mean Variance Approximations to Expected Logarithmic Utility, *Operations Research*, **31**, 4, 685–696.

Rubinstein, R. Y. (1981). *Simulation and the Monte Carlo Method* (John Wiley, New York).

Schuenemeyer, F. H., and Drew L. F. (1983). A Procedure to Estimate the Present Population of the Size of Oil and Gas Fields as Revealed by a Study of Economic Truncation, *Mathematical Geology*, **15**, 1, 145–161.