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Optimality in infinite horizon discrete time models of resource management

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We study an infinite horizon discrete time optimization problem of the Bolza type. It is argued that this problem arises frequently in models of resource management. We obtain a characterization of optimality which is an analog to the Euler equation. The results extend those of Rockafellar and Wets (1981). Furthermore, we make no assumption about free disposal and absorbing states.

1. Introduction

In this paper we study the following infinite horizon problem

$$(P) \text{ minimize } L(x)=\sum_{t=1}^{\infty} L_t(x_{t-1}, \Delta x_t)$$

over all bounded sequences $x=(x_t)_{t=0}^{\infty}$ in R^n.

Here $\Delta x_t = x_t - x_{t-1}$ and L_t: $R^n \to (-\infty, \infty]$, $t=1, 2, \ldots$ are lower semicontinuous convex functions, none of which is identically $+\infty$. We assume that for every bounded sequence $(x_t)_{t=0}^{\infty}$ in R^n, $(L_t(x_{t-1}, \Delta x_t))_{t=1}^{\infty}$ majorizes at least one absolutely convergent sequence in R^1. For this reason we adopt the convention that $L(x)=+\infty$ when the series $\sum_{t=1}^{\infty} L_t(x_{t-1}, \Delta x_t)$ does not converge. In order to exclude pathological cases we will also assume that $\inf L(x) > -\infty$.

We emphasize that L and L_t, $t=1, 2, \ldots$ are convex functions into the extended real line. The cost $+\infty$ represents the fact that certain trajectories are impossible or forbidden.

Specifically, $L_t(x_{t-1}, \Delta x_t)=+\infty$ means that the pair x_{t-1}, x_t violates some implicit constraints. For example, L_t may be defined by

$$L_t(x_{t-1}, \Delta x_t)=\begin{cases} +\infty & \text{if } (x_{t-1}, x_t)\notin C_t \\ l_t(x_{t-1}, \Delta x_t) & \text{otherwise} \end{cases}$$

If C_t is a convex set and l_t is a convex function then L_t is a convex function. The fisheries model in § 2 will make this point more clear.

Note that (P) resembles problems of Bolza type in the classical calculus of variations, the difference being that time is now discrete and the horizon is infinite. Recall that perturbations and partial integration play a vital role in the derivation of the Euler equation. Therefore it should come as no big surprise that in the discrete case perturbations and partial summation come to the front stage.

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The paper is outlined as follows. Section 2 describes the importance of the problem in resource management. For concreteness, we illustrate by means of a standard model of fisheries management. Section 3 deals with a strong characterization of optimality in terms of dual variables. In § 4 we demonstrate that the latter variables arise as a solution to a dual optimization problem. These results have all been obtained in the finite horizon case by Rockafellar and Wets (1981). The final section provides interpretation of the results.

2. A fisheries model

The importance of (P) in resource management, where a long run dynamic perspective is required, cannot be overstated. As an example consider the following. Let $c_{t-1} \in R_+^n$ be the catch of fish and let $x_{t-1} \in R_+^n$ be the escapement after catch at the beginning of period $t-1$. Here R_+^n denotes the set of all n-vectors with non-negative coordinates. The dynamics are governed by the equation

$$x_t = f_{t-1}(x_{t-1}) - c_t, \quad t \geqslant 1$$

and the profit in period t is $\pi_t(x_{t-1}, c_t)$.

Let δ, $0<\delta<1$ be a discount fact and let the initial point $\bar{x}_0 \geqslant 0$ be given. The objective is to maximize the present value

$$\sum_{t=1}^{\infty} \delta^t \pi_t(x_{t-1}, c_t)$$

We translate this optimization problem into the form (P) by defining

$$L_1(x_0, \Delta x_1) = \begin{cases} -\delta\pi_1(x_0, f_0(x_0) - x_1) & \text{if } x_0 = \bar{x}_0,\ x_1 \geqslant 0 \quad \text{and} \quad x_1 \leqslant f_0(x_0) \\ +\infty & \text{otherwise} \end{cases}$$

and for $t>1$

$$L_t(x_{t-1}, \Delta x_t) = \begin{cases} -\delta^t \pi_t(x_{t-1}, f_{t-1}(x_{t-1}) - x_t) & \text{if } x_{t-1}, x_t \geqslant 0,\ x_t \leqslant f_{t-1}(x_{t-1}) \\ +\infty & \text{otherwise} \end{cases}$$

Assume that π_t and f_{t-1} are both concave and continuous for all $t \geqslant 1$. Also suppose that $\pi_t(x_{t-1}, c_t)$ is monotonically non-decreasing in c_t. Then it is straightforward to demonstrate that L_t is convex and lower semi-continuous.

3. Characterization of optimality

The purpose of this section is to demonstrate strong (necessary and sufficient) duality relations. After defining some notations we give in Theorem 1 a sufficient condition for optimality in terms of dual variables. For a necessary condition an assumption about strict feasibility will be made (a 'Slater' condition). When this condition is in force we are able to demonstrate (Theorem 2) that dual variables exist in a neat form.

For notational convenience denote by l_n^∞ the set of bounded sequences in R^n, and let l_n^1 be the set of sequences $(p_t)_{t=1}^\infty$ in R^n such that $\sum_{t=1}^{\infty} |p_t| < \infty$.

Both these spaces are so-called Lebesgue spaces. They are in fact Banach spaces and l_n^∞ is the dual of l_n^1 under the natural pairing

$$\sum_{t=1}^{\infty} p_t \cdot y_t$$

Now define

$$\phi: l_n^\infty \to [-\infty, \infty] \text{ by } \phi(y) = \inf_{x \in l_n^\infty} \sum_{t=1}^{\infty} L_t(x_{t-1}, \Delta x_t + y_t)$$

Clearly $\phi(y)$ is convex. Observe that $y \in l_n^\infty$ plays the role of a perturbation. In general denote by $\partial g(z)$ the set of subgradients of g at z.

Theorem 1

Let $x \in l_n^\infty$ and $p \in l_n^1$. Then x is an optimal solution of (P) and $p \in \partial\phi(0)$ if and only if $(\Delta p_t, p_t) \in \partial L_t(x_{t-1}, \Delta x_t)$ for all $t \geqslant 1$ where $\Delta p_t = p_t - p_{t-1}$.

Proof

$p \in \partial\phi(0)$ means that $\phi(0) \leqslant \phi(y) - py$ for all $y \in l_n^\infty$. Consequently x solves (P) and $p \in \partial\phi(0)$ if and only if

$$\sum_{t=1}^{\infty} L_t(x'_{t-1}, \Delta x'_t + y_t) - \sum_{t=1}^{\infty} p_t \cdot y_t \tag{1}$$

attains its minimum at $x' = x$ and $y = 0$.

Now define $z_t = x'_{t-1}$ and $w_t = \Delta x'_t + y_t$ for $t = 1, 2, \ldots$

Let $p_0 = 0$. Then

$$\sum_{t=1}^{T} p_t \cdot y_t = \sum_{t=1}^{T} p_t \cdot (w_t - \Delta x'_t)$$

$$= \sum_{t=1}^{T} p_t \cdot w_t - p_T \cdot x'_T + \sum_{t=1}^{T} \Delta p_t \cdot x'_{t-1}$$

Here $p_T \cdot x'_T \to 0$ when $T \to \infty$. Hence after this change of variables (1) becomes

$$\sum_{t=1}^{\infty} \{L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t \cdot w_t\} \tag{2}$$

We are to minimize (2) over all $z, w \in l_n^\infty$. Note that the criterion (2) is separable.

Furthermore, $L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t \cdot w_t$ attains its minimum at $(x_{t-1}, \Delta x_t)$ if and only if $(\Delta p_t, p_t) \in \partial L_t(x_{t-1}, \Delta x_t)$. This completes the proof. Q.E.D.

Remark 1

The significance of this result is that for purposes of computation or planning, the dual variables allow for the optimization problem (P) to be completely decomposed into an infinite sequence of independent subproblems. In its original version any perturbation of x_{t-1} affects not only $L_t(x_{t-1}, \Delta x_t)$ but also $L_s(x_{s-1}, \Delta x_s)$ for all $s > t$. By contrast, suppose L_t is strictly convex. Then the choice of z_t, w_t so as to minimize $L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t \cdot w_t$ is completely free from all $t \geqslant 1$ in the sense that it has no future effects. In general, when $L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t \cdot w_t$ does not have a unique minimum (z_t, w_t) we must check the feasibility requirement: $z_t + w_t = z_{t+1}$.

Remark 2

The dual variables $p=(p_t)_{t=1}^{\infty}$ may be interpreted as follows. Fix a price p_t for perturbing the rate of change in period t. Whenever a perturbation profile $y=(y_t)_{t=1}^{\infty}$ is chosen we incur an additional cost $py=\sum_{t=1}^{\infty} p_t \cdot y_t$.

It follows that for a given price regime p on perturbations $\phi(y)-py$ is the lowest possible cost. We say that p is an 'equilibrium price' regime if no perturbation pays off. This means that $\phi(0)\leqslant\phi(y)-p(y)$, i.e. $p\in\partial\phi(0)$. Theorem 1 says that minimum cost is achieved at x and p is an equilibrium price regime if p supports the optimal trajectory in the sense that $(\Delta p_t, p_t)\in\partial L_t(x_{t-1}, \Delta x_t)$ for all $t\geqslant 1$.

Remark 3

Theorem 1 provides a sufficient condition for optimality, i.e. if $x\in l_n^{\infty}$, $L(x)<\infty$, $p\in l_n^1$ and $(\Delta p_t, p_t)\in\partial L_t(x_{t-1}, \Delta x_t)$, for all $t\geqslant 1$, then x solves (P). We would also want a necessary characterization of optimality saying that if $x\in l_n^{\infty}$ is optimal in (P) then there exists $p\in l_n^1$ such that $(\Delta p_t, p_t)\in\partial L_t(x_{t-1}, \Delta x_t)$. For this purpose two additional questions must be addressed. First, remember that $p\in\partial\phi(0)$. Hence we must guarantee that $\partial\phi(0)$ is non-empty. Second, since ϕ is a convex functional from l_n^{∞} to $[-\infty, \infty]$ we have $\partial\phi(y)\subseteq(l_n^{\infty})^*$ at any $y\in l_n^{\infty}$. However, l_n^1 is not reflexive, i.e. $(l_n^1)^{**}=(l_n^{\infty})^*\supseteq l_n^1$ where the last inclusion is strict (Brown and Pearcy 1979). In order to have an amenable representation of $p\in\partial\phi(0)$ we want $\partial\phi(0)$ to be a subset of l_n^1. Both these questions are resolved in the following theorem.

Theorem 2

Suppose there exists $\epsilon>0$, suppose $x\in l_n^{\infty}$ is optimal and a summable sequence $(\alpha_t)_{t=1}^{\infty}$ of non-negative real numbers such that for all $t\geqslant 1$ $L_t(z, w)\leqslant\alpha_t$ when $|z-x_{t-1}|\leqslant\epsilon$, $|w-\Delta x_t|\leqslant\epsilon$. Then $\partial\phi(0)$ is non-empty. Furthermore, $\partial\phi(0)$ may be identified with a weakly compact subset of l_n^1. It follows that for every optimal solution x of (P) a $p\in l_n^1$ exists such that $(\Delta p_t, p_t)\in\partial L_t(x_{t-1}, \Delta x_t)$ for all $t\geqslant 1$.

Proof

Let $y_t\in R^n$ be such that $|y_t|\leqslant\epsilon$ for every $t\geqslant 0$. Define $x'_t=x_t-y_t$ for $t\geqslant 0$ and $z_t=x'_{t-1}$, $w_t=\Delta x'_t+y_t$ for $t\geqslant 1$. Then $z_t-x_{t-1}=-y_{t-1}$ and $w_t-\Delta x_t=y_{t-1}$. Consequently

$$\sum_{t=1}^{\infty} L_t(x'_{t-1}, \Delta x'_t+y_t)=\sum_{t=1}^{\infty} L_t(z_t, w_t)\leqslant\sum_{t=1}^{\infty}\alpha_t$$

Hence $\phi(y)<\infty$ for all $y\in l_n^{\infty}$ with $|y|\leqslant\epsilon$. It follows that $0\in\operatorname{int}\{y\,|\,\phi(y)<\infty\}$ and this suffices to guarantee that $\partial\phi(0)$ is non-empty. The next statement of the theorem follows from corollary 2c of Rockafellar (1971). The final assertion is part of Theorem 1. This completes the proof. Q.E.D.

4. Duality

In this section we show that the price profile $p\in l_n^1$ supporting an optimal trajectory may arise as the solution to a dual problem. This latter problem may be stated as

$$(D)\ \text{maximize}\ \sum_{t=1}^{\infty} L_t(\Delta p_t, p_t)\ \text{over all}\ p\in l_n^1$$

Here $L_t: R^n \times R^n \to [-\infty, \infty)$ is defined by

$$L_t(\alpha, \beta) = \inf_{a,\, b \in R^n} \{(L_t(a, b) - \alpha \cdot a - \beta \cdot b\}$$

Let inf (P), sup (D) denote the optimal value of (P) and (D) respectively.

Theorem 3

We always have inf $(P) \geqslant$ sup (D). Furthermore $p \in \partial\phi(0) \cap l_n^1$ if and only if inf $(P) =$ max (D) and p is optimal for (D).

Proof

Clearly

$$\inf (P) = \phi(0) \geqslant \inf_{y \in l_n^\infty} \{\phi(y) - py\} = \inf_{x,\, y \in l_n^\infty} \sum_{t=1}^{\infty} L_t(x_{t-1}, \Delta x_t + y_t) - py$$

$$= \inf_{z,\, w \in l_n^\infty} \sum_{t=1}^{\infty} \{L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t w_t\} \geqslant \sum_{t=1}^{\infty} L_t(\Delta p_t, P_t)$$

Here we applied the same change of variable as in the proof of Theorem 1. It follows that inf $(P) \geqslant$ sup (D) and equality holds if and only if $\phi(0) + py \leqslant \phi(y)$ for all $y \in l_n^\infty$, i.e. $p \in \partial\phi(0)$. Q.E.D.

5. Interpretation of results

We want to conclude by restating some of the preceding results in the context of the fisheries model. It goes without saying that the same results apply to models of capital growth. Let $P_t = \delta^{-t} p_t$. In this case the condition $(\Delta p_t, p_t) \in \partial L_t(x_{t-1}, \Delta x_t)$ means that

$$\delta\pi_t(x'_{t-1}, c'_t) + (\delta P_t x'_t - P_{t-1} x'_{t-1}) \leqslant \delta\pi_t(x_{t-1}, c_t) + (\delta P_t x_t - P_{t-1} x_{t-1})$$

for all x'_{t-1}, c'_t with $x'_t = f_t(x'_{t-1}) - c'_t$ and $x_t = f(x_{t-1}) - c_t$. This says of course that the sum of current profit and the capital gain, the latter being evaluated at prices $(P_t)_{t=1}^{\infty}$, is maximal along the optimal trajectory. These prices might be termed a regime of 'equilibrium' (or efficiency) prices. This word underlines the salient feature that under these prices even a short-sighted manager would not find it profitable to deviate from the long-run optimal trajectory. In fact, Theorem 1 says that if an equilibrium price regime exists (and π_t is strictly concave) then we may perform the dynamic optimization by simply maximizing the current sum of profit and capital gain in a completely myopic manner. Conversely, suppose that for some $\epsilon > 0$ and x, $c \in l_n^\infty$, that $\pi_t(x'_t, c'_t) \geqslant \beta_t$ when $|x'_t - x_t| \leqslant \epsilon$ and $|c'_t - c_t| \leqslant \epsilon$ where $\sum_{t=1}^{\infty} \delta^t |\beta_t| < \infty$. Then by Theorem 2, if $x \in l_n^\infty$ is optimal there exists an equilibrium price regime P supporting x.

6. Conclusions

We have demonstrated that optimality in some convex infinite horizon models is characterized in terms of an equilibrium price system. These prices support the optimal trajectory in such a way that the mathematical programming problem is decomposed with respect to time. Similar results have been obtained by several authors. See Weitzman (1973), Polterovich (1983), McKenzie (1976). However, it is common practice in these papers to invoke assumptions about free disposal and 'nothing ventured nothing gained' or to assume some kind of compactness. In our proof we have completely avoided such restrictions. Also our proof does not depend on any argument from the realm of dynamic programming.

We emphasize that the characterization

$$(\Delta p_t, p_t) \in \partial L_t(x_{t-1}, \Delta x_t),\ t \geq 1 \tag{3}$$

is a discrete time version of the well-known Euler equation

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}} L_t(x(t), \dot{x}(t)) = \frac{\partial}{\partial x} L_t(x(t), \dot{x}(t)) \tag{4}$$

To see this, define

$$p(t) = \frac{\partial}{\partial \dot{x}} L_t(x(t), \dot{x}(t)) \tag{5}$$

Then by (4) and (5) we get

$$(\dot{p}(t), p(t)) = \nabla L_t(x(t), \dot{x}(t)) \tag{6}$$

In the non-smooth case (6) becomes

$$(\dot{p}(t), p(t)) \in \partial L_t(x(t), \dot{x}(t)) \tag{7}$$

Now discretize (7) to get (3). We conclude by showing that in economic applications convexity is often very appropriate but differentiability assumptions are rather unnatural. To this end we elaborate slightly on the fisheries example of § 2. Let y_t be the activity vector of the fishing industry in period t. Suppose y_t is chosen so as to maximize the profit $a_t \cdot y_t$ subject to the constraints

$$A_t Y_t \leqslant b_t(x_t, x_{t+1}),\ Y_t \geq 0$$

Suppose also that each coordinate of b_t is a concave function of x_t, x_{t+1}. Then it is easy to show that the optimal value $\pi_t(x_t, x_{t+1})$ is indeed a concave function of x_t, x_{t+1}. However, we can not expect π_t, being the optimal value of a *LP*-problem, to be differentiable with respect to the data in a classical sense. This explains why the relaxation to subdifferentiability is appropriate and not only of academic interest.

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