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Properties of Pareto-opimal allocations of resources to activities

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A linear multi-objective decision problem is considered, of the maximization of the effect of allocations of resources to activities. Necessary and sufficient conditions for a feasible solution of the problem to be Pareto-optimal are derived, in terms of properties of the allocation matrix and a certain matrix of efficiency coefficients of the allocations. A condition is given for all optimal solutions to be simple in the sense that they contain a small number of non-zero allocations. A feasible change of the positive allocations of an optimal solution produces a new optimal solution.

1. Introduction

The following multi-objective resource allocation problem, denoted by M, is considered

$$M\text{: Maximize } z=(x_1, \ldots, x_K)$$

where

$$x_k = \sum_{j=1}^{J} \alpha_{jk} x_{jk}; \quad k=1, \ldots, K \tag{1}$$

subject to the constraints

$$\sum_{k=1}^{K} x_{jk} = h_j; \quad j=1, \ldots, J \tag{2}$$

$$x=(x_{jk}) \geqslant 0 \tag{3}$$

where J is the number of resources; K is the number of activities; x_{jk} is the quantity of resource j allocated to activity k; h_j is the available quantity of recourse j; and α_{jk} is the effectiveness of resource j when allocated to activity k.

It is assumed that h_j and the α_{jk} are non-negative real numbers; and that (α_{jk}) satisfies the condition

$$\forall j \exists k\colon \alpha_{jk} > 0 \tag{4}$$

By definition, an allocation matrix x satisfying the constraints (2) and (3) is a solution of M if and only if x is Pareto-optimal in the sense that, for all feasible solutions $x' = (x'_{jk})$ of M, with objective vector z' the following condition is satisfied

$$z' \geqslant z \Rightarrow z' = z$$

where the components x'_k for $k=1, \ldots, K$ of the vector $z'=(x'_1, \ldots, x'_K)$ are given from $x_{jk}=x'_{jk}$ in eqn. (1). A Pareto-optimal solution of M will, in the following, be referred to as an optimal solution of M.

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Although general and linear multi-objective optimization problems, and associated numerical solution methods, have been considered by for instance: Gal and Leberling (1981), Roy and Vincke (1981); Zionts and Wallenius (1980) and other authors, there is no paper, known to the present author, that is concerned with the special properties of the optimal solutions of the resource allocation problem M defined in this paper. Consequently it is the present purpose to derive necessary and sufficient conditions for a feasible solution x of M to be optimal, in terms of properties of the allocation matrix x and the efficiency matrix (α_{jk}). In particular, a condition is given for an optimal solution to be simple in the sense that it contains at most $(J+K-1)$ positive allocations x_{jk}; it is noted that a solution is always optimal if it contains at most three non-zero allocations; and a feasible change of the positive allocations of an optimal solution produces a new optimal solution. A numerical example is given. It is demonstrated, that an optimal solution of M is not necessarily a solution of an associated problem of the maximization of the sum of some utility returns of the activities.

Before proceeding to derive optimality conditions of the problem M, it is observed that the aim of this paper is to derive properties of optimal solutions rather than to derive solution-algorithms, as several such algorithms of a general nature may be identified by the use of the above references. However, a comparison of the efficiencies of the these general algorithms and the efficiency of a dedicated solution-algorith of the problem M is a relevant area for further investigation; the dedicated algorithm being constructed on the basis of properties of the problem M demonstrated in this paper.

2. Efficiency conditions

Define the following subset of $\{1, ..., J\}x\{1, ..., K\}$

$$A=\{(j, k) \mid \alpha_{jk}>0\}$$

and the problem MA obtained from M by the introduction of the condition

$$x_{jk}=0 \quad \text{if } (j, k)\notin A$$

The condition (4) implies the validity of the following:

Theorem 1

The sets of efficient solutions of M and MA are identical.

According to Danskin (1967) a sequence of feasible solutions of MA given by

$$x_{j_1k_1}x_{j_1k_2}, \ldots, x_{j_ik_i}x_{j_ik_{i+1}}, \ldots, x_{j_{n-1}k_{n-1}}x_{j_{n-1}k_n} \tag{5}$$

is called a closed chain or a cycle if $k_n=k_1$; the ratio ρ of such a cycle was defined by Einbu (1978) as $\rho=\gamma^{-1}$, where

$$\gamma=\prod_{i=1}^{n-1}(\alpha_{j_ik_{i+1}}\mid\alpha_{j_ik_i}) \tag{6}$$

A shift along a cycle (5) of non-zero allocations is given by the replacement of $x_{j_ik_i}$ and $x_{j_ik_{i+1}}$ by $x'_{j_ik_i}$ and $x'_{j_ik_{i+1}}$, where

$$x'_{j_ik_i}=x_{j_ik_i}-\Delta_i; \quad i=1, \ldots, (n-1) \tag{7}$$

$$x'_{j_ik_{i+1}}=x_{j_ik_{i+1}}+\Delta_i; \quad i=1, \ldots, (n-1) \tag{8}$$

$$\Delta_i\alpha_{j_ik_{i+1}}-\Delta_{i+1}\alpha_{j_{i+1}k_{i+1}}=0; \quad i=1, \ldots, (n-2) \tag{9}$$

and the Δ_i for $i=1, \ldots, (n-1)$ are chosen such that $x'_{j_ik_i} \geqslant 0$ and $x'_{j_ik_{i+1}} \geqslant 0$ for $i=1, \ldots, (n-1)$.

Defining

$$x'_{jk} = x_{jk} \quad \text{if } (j, k) \notin \bigcup_{i=1}^{n-1} \{(j_i, k_i), (j_i, k_{i+1})\} \tag{10}$$

it follows that $x' = (x'_{jk})$ is a feasible solution of MA with the x'_k of eqn. (1) satisfying

$$x'_k = x_k \quad \text{for } k \neq k_1 \tag{11}$$

$$\Delta x'_{k_1} = x'_{k_1} - x_{k_1} = \Delta_1 \alpha_{j_1k_1}[\gamma - 1] \tag{12}$$

If

$$\xi_{i-1} = \prod_{q=1}^{i-1} (\alpha_{j_{q+1}k_{q+1}} | \alpha_{j_qk_{q+1}})$$

the value of Δ_1 is chosen such that

$$\Delta_1 \leqslant \min_{i=1, \ldots, (n-1)} [x_{j_ik_i}\xi_{i-1}] = \overline{\Delta}_1 \quad \text{if } \Delta_1 > 0 \tag{13}$$

and

$$\Delta_1 \geqslant \max_{i=1, \ldots, (n-1)} [-x_{j_ik_{i+1}}\xi_{i-1}] = \underline{\Delta}_1 \quad \text{if } \Delta_1 < 0 \tag{14}$$

observing that

$$x'_{j_mk_m} = 0 \quad \text{if } \Delta_1 = \overline{\Delta}_1 \quad \text{and} \quad \overline{\Delta}_1 = x_{j_mk_m}\xi_{m-1} \tag{15}$$

and

$$x'_{j_mk_{m+1}} = 0 \quad \text{if } \Delta_1 = \underline{\Delta} \quad \text{and} \quad \underline{\Delta}_1 = -x_{j_mk_{m+1}}\xi_{m-1} \tag{16}$$

With reference to theorem 1 the main result of this paper is given as follows.

Theorem 2

A feasible solution x of MA is Pareto-optimal if and only if one of the two following conditions is satisfied: (1) there is no cycle (5) of positive allocations $x_{j_ik_i}$ and $x_{j_ik_{i+1}}$ for $i=1, \ldots, (n-1)$, or otherwise, (2) the ratio ρ of all such cycles is equal to 1.

Proof

Assume that there exists an optimal solution x of MA that contains a cycle (5) of positive allocations. If the ratio ρ of the cycle is not equal to 1, it follows that one of the x_k, say x_{k_1}, can be increased without a change of the other x_k for $k \neq k_1$. Specifically, the replacement of x by x', where the x'_{jk} are given by the eqns. (7)–(10), demands that the relationships (11) and (12) become satisfied; and $\Delta x'_{k_1} > 0$ if Δ_1 is chosen such that $\Delta_1 > 0$ if $\gamma > 1$ and $\Delta_1 < 0$ if $\gamma < 1$; noting that a $\Delta_1 \neq 0$ can be selected, subject to the requirements (13) and (14), since $x_{j_ik_i} > 0$ and $x_{j_ik_{i+1}} > 0$ for $i=1, \ldots, (n-1)$ by assumption, and ξ_{i-1} exists and is positive for $i=1, \ldots, n-1$ by the definition of the problem MA. Since $\Delta x'_{k_1} > 0$ the eqns. (11) and (12) show that x is not optimal, a contradiction, which implies that $\rho = 1$ is the only possibility, since $\rho = \gamma^{-1}$.

Conversely, let the condition (1) or the condition (2) of the theorem be satisfied. Assuming, by contradiction, that x is not optimal, there exists a feasible solution $y=(y_{jk})$ of MA such that

$$y_{k_1} > x_{k_1} \tag{17}$$

and

$$y_k = x_k \quad \text{for all } k \neq k_1 \tag{18}$$

where y_k is obtained from eqn. (1) with (x_{jk}) replaced by (y_{jk}).

This follows from the observation that if $y_k > x_k$ for a $k \neq k_1$ then at least one of the y_{jk} can be reduced with an increase of y_{jk_1}, until y_k becomes equal to x_k.

Equation (17) requires the existence of a $j_1 \in \{1, \ldots, J\}$ such that

$$y_{j_1k_1} > x_{j_1k_1}$$

which, in combination with the constraint (2) for $j=j_1$, implies that

$$y_{j_1k_2} < x_{j_1k_2}$$

for some $k_2 \neq k_1$. The eqns. (1) and (18) for $k=k_2$ demand that there exists a $j_2 \neq j_1$ such that

$$y_{j_2k_2} > x_{j_2k_2}$$

Continuation of the above arguments demands the existence of a cycle of allocations given by

$$y_{j_mk_m}, y_{j_mk_{m+1}}, \ldots, y_{j_{n-1}k_{n-1}}, y_{j_{n-1}k_n} \tag{19}$$

if a previously selected activity k_m is repeated: $k_n = k_m$; a similar cycle results if a previously selected resource is repeated, and a repetition must occur since there is a finite number of resources and activities. In both cases, the cycle can, for notational convenience, and without loss of generality, be represented in the form (19) with $m=1$.

Since $y_{j_ik_i} > x_{j_ik_i} \geqslant 0$ a cyclic shift, given by the eqns. (7)–(10) with x_{jk} replaced by y_{jk} for all (j, k) and $\Delta_1 > 0$, can be applied to produce a cycle of positive allocations given by

$$x'_{j_1k_1}\ x'_{j_1k_2}, \ldots, x'_{j_{n-1}k_{n-1}}\ x'_{j_{n-1}k_n} \tag{20}$$

The implication is that the condition 2 of the theorem is satisfied, demanding that the ratio ρ' of the cycle (20) is equal to 1, and the eqns. (11)–(12) show that

$$x'_k = y_k \quad \text{for } k=1, \ldots, K \tag{21}$$

since $\gamma' = \rho' = 1$.

An extension of the arguments leading to the eqns. (11)–(16) shows that the value of Δ_1 can be chosen to produce a feasible allocation matrix of MA, denoted by $x^1 = (x_{jk}{}^1)$ (rather than $x' = (x'_{jk})$), such that for $i=1, \ldots, n-1$:

$$x_{j_ik_i}{}^1 \geqslant x_{j_ik_i} \tag{22}$$

$$x_{j_ik_{i+1}}{}^1 \leqslant x_{j_ik_{i+1}} \tag{23}$$

and with at least one of the above inequalities being satisfied as an equality.

Since $x_{jk}{}^1 = y_{jk}$ if $(j, k) \notin \{(j_i, k_i), (j_i, k_{i+1})\}$ for all $i = 1, \ldots, (n-1)$ by eqn. (10) it follows that the number of (j, k) for which $x_{jk}{}^1 \neq x_{jk}$ is strictly smaller than the number of (j, k) for which $y_{jk} \neq x_{jk}$. Since by the eqns. (17), (18) and (21)

$$x_{k_1}{}^1 > x_{k_1} \tag{24}$$

$$x_k{}^1 = x_k \quad \text{for } k \neq k_1 \tag{25}$$

the previously given arguments can be repeated with y replaced by x^1 to demonstrate the existence of an $x^2 = (x_{jk}{}^2)$, satisfying the relations (22)–(25) with x^1 replaced by x^2, such that the number of (j, k) with $x_{jk}{}^2 \neq x_{jk}$ is smaller than the number of (j, k) with $x_{jk}{}^1 \neq x_{jk}$ and such that $x_k{}^2 = y_k$ for $k = 1, \ldots, K$. Since there is a finite number of different ordered pairs (j, k), repetition of the previously given arguments must lead to an allocation x^N such that $x^N = x$ and $x_{k_1}{}^N = x_{k_1}{}^{N-1} = \ldots = x_{k_1}{}^1 = y_{k_1} > x_{k_1}$, a contradiction, which demands that x is optimal. q.e.d.

The theorem implies the validity of the following.

Corollary 1

If the efficiency matrix (α_{jk}) contains no cycle with ratio 1, then any optimal solution x of MA contains at most $(J+K-1)$ positive allocations x_{jk}.

Proof

An allocation matrix $x = (x_{jk})$ with at least $(J+K)$ positive allocations x_{jk} must contain a cycle. q.e.d.

The following result is of particular interest for problems with $J = 1$.

Corollary 2

A feasible solution x of MA is optimal if it contains at most three positive allocations.

Corollary 3

If x is an optimal solution of MA and the set S is defined by

$$S = \{(j, k) \mid x_{jk} > 0\}$$

then any feasible solution x' of MA such that:

$$x'_{jk} = 0 \quad \text{if } (j, k) \notin S$$

is an optimal solution of MA.

3. A numerical example

The data of a particular problem MA is given by:

$$J = K = 3$$

$$(\alpha_{jk}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$(h_j) = (3, 3, 3)$$

The feasible solution

$$x=\begin{pmatrix}1 & 2 & 0\\ \delta_1 & (3-\delta_1-\delta_2) & \delta_2\\ 0 & 1 & 2\end{pmatrix}$$

is optimal if $\delta_1=\delta_2=0$, since it contains no cycle in this case. The solution is not optimal if $\delta_2\neq 0$ and $(\delta_1+\delta_2)<3$, since there is a cycle with ratio 2 of positive allocations.

However, is $\delta_2=0$ the solution is optimal for all $\delta_1\in[0, 3]$; furthermore, for any such δ_1 optimal solutions can be generated by a cyclic shift, defined by the eqns. (7)–(10), along the cycle $x_{11}x_{12}x_{22}x_{21}$. Additionally, the positive allocations may be changed (with $x_{13}=x_{31}=0$) as long as feasibility of MA is maintained, according to Corollary 3.

4. Return maximization

Luss and Gupta (1975), Einbu (1978), Mjelde (1977), Mjelde (1983) and other authors have considered the resource allocation problem P of the maximization of

$$z=\sum_{k=1}^{K} r_k(x_k)$$

with x_k given by eqn. (1) for $k=1, \ldots, K$, subject to the constraints (2) and (3) of M. The r_k for $k=1, \ldots, K$ are continuously differentiable, strictly increasing and strictly concave functions with $r_k(0)=0$, describing the utility of the returns from the activities due to given allocations.

Einbu (1978) applied properties of the pattern of non-zero allocations in a solution matrix (x_{jk}) of P and induction on the number of rows of (x_{jk}), to demonstrate a result analogous to that of Corollary 1 for the problem P; and Mjelde (1977) demonstrated this result for the problem P by the application of an associated linear programming problem with a set of optimal solutions identical to the set of optimal solutions of the problem P. A resource allocation problem R with a fractional objective function given by the return of the problem P divided by an affine cost function was considered by Mjelde (1981), on the basis of arguments analogous to those applied in the proof of Theorem 2 of this paper.

It is now relevant to ask the question if any optimal solution of MA is also an optimal solution of a problem P for some return functions r_k. The answer to this question is no, as can be seen from the following example of a problem M with

$$J=K=2$$

$$(\alpha_{jk})=\begin{pmatrix}1 & 4\\ 4 & 2\end{pmatrix}$$

and

$$(h_j)=(1, 1)$$

The solution

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is optimal. Assuming that there exists some strictly increasing return functions r_k of P with derivatives r'_k such that x is optimal for P, it follows, from Kuhn–Tucker optimality conditions of P, that there exists real numbers $\lambda_j \geqslant 0$ for $j = 1, 2$ such that:

$$r'_1(1) \cdot 1 = \lambda_1; \quad r'_2(1) \cdot 4 \leqslant \lambda_1$$

$$r'_1(1) \cdot 4 \leqslant \lambda_2; \quad r'_2(1) \cdot 2 = \lambda_2$$

which implies that

$$4\lambda_1 \leqslant \lambda_2; \quad 2\lambda_2 \leqslant \lambda_1$$

and consequently that

$$\lambda_1 = \lambda_2 = 0$$

a contradiction, since $r'_1(1) > 0$ and $r'_2(1) > 0$.

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