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A war of area attrition and aimed attack-properties of optimal strategies

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A war is considered of the attrition of the enemy's weapons supply system or the attack of his forces in the battle area. The rate of weapons supply of each side depends on the total number of successful attrition allocations of the other side; this dependency is described by a decreasing and differentiable function. The attack allocations are aimed at specific targets, while the attrition allocations are randomly distributed over the area in which the supply operations take place. The theory of differential games is applied to derive properties of optimal solutions.

1. Introduction

A differential game theoretic model of a war of prescribed duration is considered, each side allocates weapons to the attrition of the enemy's weapons supply system or to the attack of his forces in the battle field; the objective of the attack allocations is to change the rate of movement of the front line or to inflict casualties. Such models have been considered by Isaacs (1965), Berkovitz and Dresher (1960, 1957), Fulkerson and Johnson (1957) and Mjelde (1980). Several authors have analysed the problem in the context of the determination of an optimal distribution of supporting fire (artillery) between the enemy's primary forces (infantry) and his supporting forces. Taylor (1978) develops results for a Lanchester-type differential game; some other works are mentioned: Bellman and Dreyfus (1958); Giamboni, Mengel and Dishington (1951), Kawara (1973) and Weiss (1959). Mjelde (1982) considered the problem of the defence of a valuable target against enemy attacks over a certain period of time, until reinforcements arrive; using control theoretical arguments.

Mjelde (1980) introduced a rate of supply of (supporting) weapons of each side that decreased when the number of successful attrition allocations of the other side increased. The attrition and attack allocations were assumed to correspond to aimed fire, with an effectiveness proportional to the rate of fire. In this paper the attrition fire is randomly distributed over the area where the supply activities take place; the number of successful attrition allocations of each side increases at a rate proportional to the product of the rates of weapons supply of each side. The attack allocations, however, are aimed at selected targets.

It is demonstrated that an optimal solution concentrates all fire either to attrition or to attack, and that there is at most one transition from attrition to attack allocations; the war ends with all fire applied to attack. A condition is given for each side to start the battle with attrition allocations.

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2. The model

The model considered in this paper is denoted by P and defined as follows:

$$P\colon V=\min_{\phi_1}\ \max_{\phi_2}\ \int_0^T [r_2 g_2(x_2)(1-\phi_2)-r_1 g_1(x_1)(1-\phi_1)]\,dt$$

subject to

$$\dot{x}_1=\gamma_1 g_1(x_1) g_2(x_2)\phi_2 \tag{1}$$

$$\dot{x}_2=\gamma_2 g_1(x_1) g_2(x_2)\phi_1 \tag{2}$$

$$0\leqslant \phi_i\leqslant 1 \quad \text{for } i=1,2$$

$$x_i(0)=0 \quad \text{for } i=1,2$$

The $x_i(t)$ and $\phi_i(t)$ for $i=1, 2$ are functions of the time $t\geqslant 0$ and $\dot{x}_i=dx_i|dt$. The γ_i and the r_i for $i=1, 2$ are positive real constants and the g_i for $i=1, 2$ are continuously differentiable functions with $g_i(x_i)>0$ and $g'_i(x_i)=dg_i(x_i)|dx_i<0$ for all $x_i\geqslant 0$.

Defining

$$i^*=\begin{cases}1 \text{ if } i=2\\ 2 \text{ if } i=1\end{cases}$$

the quantities of the model are given as follows:

$x_i(t)$: the total number of attrition allocations inflicted on side i by side i^* in the time interval $[0, t]$.

$g_i(x_i)$: the rate of weapons supply of side i as a function of $x_i(t)$.

$\phi_i(t)$: the fraction of $g_i(x_i)$ allocated by side i to attrition at time t; the remaining weapons are allocated to attack.

γ_i: the effectiveness of an attrition allocation by side i^*.

r_i: the effectiveness of an attack allocation by side i.

T: the duration of the war.

3. Optimality conditions

If V_i denotes the partial derivative of V with respect to x_i for $i=1, 2$, the main equation of differential game theory, see Isaacs (1965) and Friedman (1971), shows that optimal solution(s) satisfy:

$$\min_{\phi_1}\ \max_{\phi_2}\ \{-C_3+r_2 g_2(x_2)-r_1 g_1(x_1)+\phi_1 g_1(x_1) S_1+\phi_2 g_2(x_2) S_2\}=0$$

for some constant C_3 and

$$S_1=V_2\gamma_2 g_2(x_2)+r_1 \tag{3}$$

$$S_2=V_1\gamma_1 g_1(x_1)-r_2 \tag{4}$$

Any optimal solution of P satisfy:

$$\phi_1=\begin{cases}0 & \text{if } S_1\geqslant 0\\ 1 & \text{if } S_1<0\end{cases};\quad \phi_2=\begin{cases}0 & \text{if } S_2\leqslant 0\\ 1 & \text{if } S_2>0\end{cases} \tag{5}$$

If

$$\tau = T - t$$

denotes the retrogressive time and

$$\mathring{V}_i = dV_i/d\tau \quad \text{and} \quad \mathring{S}_i = dS_i/d\tau$$

the path equations are given by:

$$\mathring{V}_1 = -r_1 g'_1(x_1) + \phi_1 g'_1(x_1) S_1 + \phi_2 g'_1(x_1) g_2(x_2) V_1 \gamma_1 \tag{6}$$

$$\mathring{V}_2 = r_2 g'_2(x_2) + \phi_2 g'_2(x_2) S_2 + \phi_1 g_1(x_1) g'_2(x_2) V_2 \gamma_2 \tag{7}$$

with the initial conditions:

$$V_1 = V_2 = 0 \quad \text{for } \tau = 0$$

The differentiation of the S_i given by eqns. (3) and (4) with respect to τ gives:

$$\mathring{S}_1 = \gamma_2 g_2(x_2) g'_2(x_2)(r_2 + \phi_2 S_2) + V_2 \gamma_2 g'_2(x_2)[\gamma_2 g_1(x_1) g_2(x_2)\phi_1 + \mathring{x}_2]$$

$$\mathring{S}_2 = \gamma_1 g_1(x_1) g'_1(x_1)(-r_1 + \phi_1 S_1) + V_1 \gamma_1 g'_1(x_1)[\gamma_1 g_1(x_1) g_2(x_2)\phi_2 + \mathring{x}_1]$$

where the expressions inside the square brackets are both equal to zero, because of eqns. (1) and (2) and the relationship $\mathring{x}_1 = -\dot{x}_i$ (which follows from $\tau = T - t$). The implication is that:

$$\mathring{S}_1 = \gamma_2 g_2(x_2) g'_2(x_2)[r_2 + \phi_2 S_2] \tag{8}$$

$$\mathring{S}_2 = \gamma_1 g_1(x_1) g'_1(x_1)[-r_1 + \phi_1 S_1] \tag{9}$$

with the initial conditions:

$$S_1 = r_1 \text{ and } S_2 = -r_2 \quad \text{for } \tau = 0 \tag{10}$$

Since $\phi_2 S_2 \geqslant 0$ and $\phi_1 S_1 \leqslant 0$ the following result is valid:

Lemma 1

$$\mathring{S}_1(\tau) < 0, \quad \mathring{S}_2(\tau) > 0 \quad \text{for } \tau \in [0, T].$$

4. Properties of optimal solutions

An optimal solution of P is denoted by $\phi_i^*(t)$ for $i = 1, 2$.

Equations (5) and (10) show that $\phi_1^* = \phi_2^* = 0$ for $\tau = 0$.

Integration of eqns. (8) and (9) with the initial condition (10) and $\phi_1^* = \phi_2^* = 0$ demands that ϕ_1^* or ϕ_2^* changes from 0 to 1 at times $\tau_1(s_2)$ or $\tau_2(s_1)$, where

$$\tau_1(s_2) = -r_1/r_2 \gamma_2 g'_2(s_2) g_2(s_2)$$

$$\tau_2(s_1) - -r_2/r_1 \gamma_1 g'_1(s_1) g_1(s_1)$$

and

$$s_i = x_i(T) \quad \text{for } i = 1, 2$$

Define:

$$\tau_{10} = \text{Min } \tau_1(s_2) \quad \text{for } 0 \leqslant s_2 \leqslant \gamma_2 g_1(0) g_2(0) T$$

$$\tau_{20} = \text{Min } \tau_2(s_1) \quad \text{for } 0 \leqslant s_1 \leqslant \gamma_1 g_1(0) g_2(0) T$$

Lemma 1 and the previously given observations give:

Theorem 1

Any optimal solution has the following properties:

(1) $\phi_1^* = \phi_2^* = 0$ for $\tau \leqslant \mathrm{Min}\,(\tau_{10}, \tau_{20})$

(2) $\phi_i^* \in \{0, 1\}$ for $i = 1, 2$ and $\phi_i^*(t)$ changes from 1 to 0 at most once (in forward time).

Figure 1 illustrates the simultaneous development of (S_1, S_2) and the associated changes of (ϕ_1, ϕ_2).

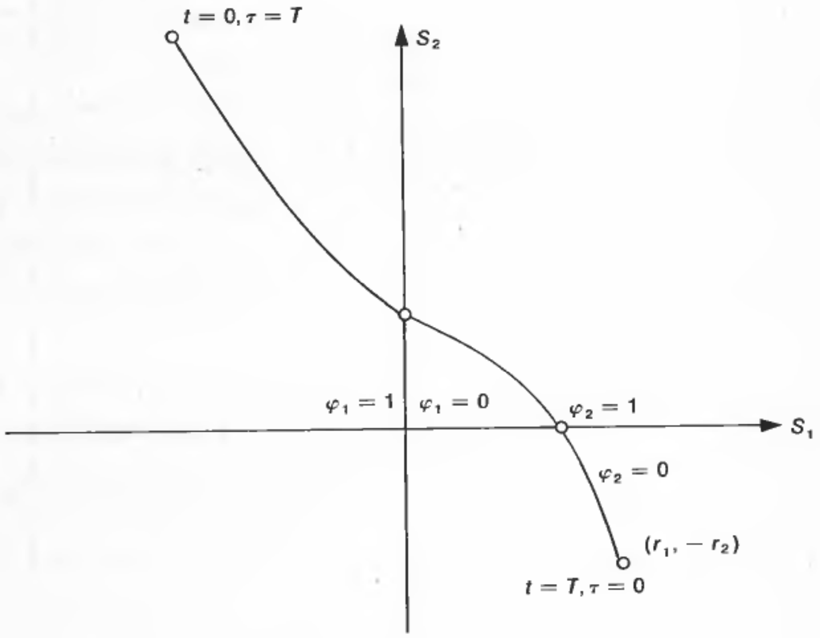

Figure 1. The (S_1, S_2) curve.

Theorem 2

If $T > \tau_i(0)$ then $\phi_i^*(0) = 1$ for $i = 1, 2$.

Proof

Consider the case $i = 1$ and assume, by contradiction, that $\phi_1^*(0) = 0$. Theorem 1 demands that $\phi_1^*(t) = 0$ for $t \in [0, T]$ and eqn. (1) that $x_2^*(t) = 0$. Since $(r_2 + \phi_2^* S_2^*) > r_2$ eqn. (8) requires that $S_1^*(t) \leqslant \bar{S}_1(t)$ where $\bar{S}_1(t)$ is given by:

$$\dot{\bar{S}}_1 = \gamma_2 g_2(0) g_2'(0) r_2$$

and

$$\bar{S}_1(0) = r_1$$

Since $\bar{S}_1(\tau) = 0$ for $\tau = \tau_1(0)$ and $T > \tau_1(0)$ it follows that $\phi_1^*(0) = 1$. q.e.d.

5. Final remarks

Let x_i^* and S_i^* denote the values of x_i and S_i corresponding to optimal allocations $\phi_i = \phi_i^*$ for $i=1, 2$. Any given lower and upper bounds of $\phi_i^*(t)$ for $t \in [0, T]$ and $i=1, 2$ (for instance defined by $0 \leqslant \phi_i^*(t) \leqslant 1$), can be used to derive bounds of $x_i^*(t)$ for $t \in (0, T]$ and $i=1, 2$ by the application of equations and arguments analogous to those of Mjelde (1982). The bounds of ϕ_i^* and x_i^* can be used to derive bounds of S_i^*, and corresponding new bounds of ϕ_i^*, and this process can be continued. These bounds are useful in the solution of the problem P by numerical methods that require initial estimates of the quantities ϕ_i^*, x_i^* or S_i^*, see for instance Bryson and Ho (1969).

Problems for future work are the analysis of models with several types of weapons on each side. Other extensions are given for the introduction of a criterion that terminates the war, for instance if side 1 wins for $V = \underline{V}$ and side 2 wins for $V = \overline{V}$, where $\underline{V} < \overline{V}$.

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