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The defence of a valuable target—a control theoretical analysis

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A problem is considered of the defence of a valuable target against enemy attacks, such as to minimize the total number of successful attacks during a given period of time. Defence weapons are allocated to:

(1) The attack of approaching enemy combat forces, or

(2) attrition of his weapon-supply systems.

A control theoretical formulation of the problem is given. Properties of optimal allocations are derived and criteria for the optimality of pure defence allocations of the type (1) given above.

1. Introduction

A military scenario is analysed, where a valuable target of side 1 is defended against attacks from side 2, side 1 allocates weapons to:

(1) the defence against approaching enemy combat forces, or

(2) the attrition of the enemy's supply systems.

The objective of the allocations is to minimize the total number of successful enemy attacks against the valuable target during a given period of time, so that the target will be capable of receiving reinforcements at the end of this period. As an example, the valuable target may be an airbase, which is defended against attacks from a given number of enemy airbases; the defender allocates aircraft to two missions:

(1) the attack of enemy aircraft in the air, or

(2) the attack of enemy airbases.

Another example is given by the defence of an important strategical position against enemy infantry and artillery attacks; the artillery of the defender is allocated against:

(1) enemy infantry, or

(2) artillery.

The purpose of this work is to derive general properties of optimal allocations, including bounds of the optimal solution and sufficient conditions for the optimality of pure defence allocations.

The problem is formulated and analysed in terms of the theory of optimal control. The weapons' supply rates of the two sides decrease according to non-linear functions of:

(1) the number of successful enemy attacks against the valuable target of the defender, or

(2) the number of attrition allocations against the attacker.

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Although the problem involves a finite number of weapons and is discrete in nature, it is analysed by the application of a continuous model; the main interest being to obtain insight related to the nature of optimal strategies, rather than to derive detailed allocation plans for the weapons. A similar approach was applied by Taylor (1978) in the analysis of fire-support strategies, using a two-sided continuous differential game theoretical model of the Lanchester (1916) type. Isaacs (1965) analysed battles of attrition and attack by game theoretical arguments; his models are the continuous counterparts of the discrete models of Berkovitz and Dresher (1960) and of Fulkerson and Johnson (1957). Mjelde (1980) introduced engagement rates of each side that decreased according to a differentiable function of a continuously varying number of enemy attacks.

The present paper first gives a formulation of the problem and derives optimality conditions and properties of optimal solutions. Upper and lower bounds of the optimal solution are obtained and applied in the derivation of a criterion for the optimality of a pure defence strategy; another criterion is derived by direct consideration of the necessary optimality conditions. In the concluding section of the paper an analytically solvable example is given with linearly decreasing weapon supply rates. Numerical solution methods are indicated.

2. The problem

The following problem, denoted by P, is considered:

$$P: \quad z = \min_{(\phi_i)} x(T)$$

subject to the differential equations:

$$\dot{x} = \sum_{i=1}^{I} \gamma_i h_i(y_i) - \beta g(x) \left(1 - \sum_{i=1}^{I} \phi_i \right) \tag{1}$$

$$\dot{y}_i = \delta_i g(x) \phi_i; \quad i = 1, \ldots, I \tag{2}$$

where

$$\sum_{i=1}^{I} \phi_i \leqslant 1 \tag{3}$$

$$0 \leqslant \phi_i \leqslant 1; \quad i = 1, \ldots, I$$

and the initial conditions are:

$$x(0) = 0 \tag{4}$$

$$y_i(0) = 0; \quad i = 1, \ldots, I \tag{5}$$

x, y_i and ϕ_i for $i = 1, \ldots, I$ are functions of the time t for $t \in [0, T]$; β, γ_i and δ_i are non-negative numbers; g and h_i are non-negative, strictly decreasing and twice differentiable functions defined for non-negative arguments. The notations $\dot{x}$ and $\dot{y}_i$ refer to the derivatives with respect to t; g', g'' and h'_i, h''_i are the first and second derivatives of g and h_i.

It is assumed that $x > 0$ for all $t \geqslant 0$, which is the case if

$$\sum_{=1}^{I} \gamma_i h_i(\delta_i g(0) T) - \beta g(0) > 0$$

The parameters of the model are defined as follows:

I: The number of weapon-supply systems of the attacker.

x: The total number of successful attacks against the valuable target of the defender.

y_i: The total number of successful attacks by the defender against the ith weapon-supply system of the attacker.

$g(x)$: The rate of weapons' supply of the defender as a function of x.

$h_i(y_i)$: The rate of weapons' supply of the ith weapon-supply system of the attacker as a function of y_i.

ϕ_i: The fraction of weapons allocated by the defender against the ith weapon-supply system of the attacker, the fraction

$$\left(1-\sum_{i=1}^{I}\phi_i\right)$$

is allocated to defence.

δ_i: The effectiveness of the defender's attack allocations.

γ_i: The effectiveness of the weapons supplied by the ith system of the attacker.

β: The effectiveness of the defence.

3. Optimality conditions

Let V and V_i for $i=1, .., I$ denote the dual (adjoint) variables of x and y_i for $i=1, ..., I$. Writing the objective function of P in the form

$$z=\min_{(\phi_i)}\int_0^T \dot{x}(t)\,dt$$

it follows that the Hamiltonian is given by:

$$H=\dot{x}+V\dot{x}+\sum_{i=1}^{I} V_i\dot{y}_i$$

The substitution of eqns. (1) and (2) demands that

$$H=(V+1)\left[\sum_{i=1}^{I}\gamma_i h_i(y_i)-\beta g(x)\right]+g(x)\sum_{i=1}^{I}\phi_i S_i$$

where

$$S_i=\beta(V+1)+\delta_i V_i; \quad i=1, ..., I$$

Since S_i appears as a factor of ϕ_i in the Hamiltonian it is convenient to work with the dual variables V and S_i for $i=1, ..., I$, these variables determine V_i uniquely from the definition of S_i.

Since H is not an explicit function of the time t the optimal solution satisfies

$$\min_{(\phi_i)} H=c_0 \tag{6}$$

for a constant c_0.

Equation (6) implies that

$$\phi_i \begin{cases} =0 & \text{if } S_i>0 \\ >0 & \text{only if } S_i=\min\limits_m S_m \end{cases}$$

showing that S_i can be interpreted as the negative effectiveness of the attrition allocation $\phi_i>0$; the smaller S_i is, the larger is the effectiveness of ϕ_i, if $S_i>0$, then $\phi_i=0$. This interpretation is related to the facts that

$$V_i=\frac{\partial z}{\partial y_i}\leqslant 0$$

and

$$V=\frac{\partial z}{\partial x}\geqslant 0$$

showing that the term $\delta_i V_i$ of S_i measures the effectiveness of the attrition allocations, while the term $\beta(V+1)$ is the defence effectiveness, the addend 1 of V indicating that the objective is to minimize $x(T)$.

Defining the retrogressive time

$$\tau=T-t$$

the necessary optimality conditions can be expressed in terms of V and S_i as follows:

$$\frac{\partial V}{\partial \tau}=-(V+1)\beta g'(x)+g'(x)\sum_{i=1}^{I}\phi_i S_i \tag{7}$$

$$\frac{\partial S_i}{\partial \tau}=\frac{\partial V}{\partial \tau}+(V+1)\delta_i\gamma_i h'_i(y_i);\quad i=1,\ldots,I \tag{8}$$

with the initial conditions:

$$V(0)=0;\quad S_i(0)=\beta;\quad i=1,\ldots,I \tag{9}$$

4. Properties of optimal solutions

Equations (6) and (9) demand the validity of:

Theorem 1

Any optimal solution satisfies $\phi_i=0$ for $i=1,\ldots,I$ for $\tau=0$.

The implication is that the battle ends with all defence weapons allocated to attack the approaching enemy combat forces, it is optimal to defend the target rather than to reduce the enemy's supply rate.

The following result will be used:

Lemma 1

$$\frac{\partial V}{\partial \tau}\geqslant 0,\quad V\geqslant 0$$

Proof

Since $S_i(0)=\beta$, it follows that $S_i(\tau)>0$ and $\phi_i=0$ for $i=1,\ldots,I$ and for τ sufficiently close to $\tau=0$, which, in conjunction with eqns. (7) and $V(0)=0$, imply that $\partial V/\partial\tau > 0$ and consequently $V \geqslant 0$ near $\tau = 0$. Since $\phi_i S_i \leqslant 0$ for $i = 1, \ldots, I$ by eqn. (6) it follows from eqn. (7) that $\partial V/\partial\tau \geqslant 0$ and $V \geqslant 0$ for all τ (since otherwise there would exist a $\tau_0 > 0$, such that $V(\tau) > 0$ and $\partial V/\partial\tau > 0$ for $\tau \in [0, \tau_0]$ and $V(\tau_0) = 0$), q.e.d.

For a $\tau \in [0, T]$ define:

$$M(\tau) = \left\{ i \mid S_i(\tau) = \min_m S_m(\tau) \right\}$$

Lemma 2

$M(\tau_1) \subset M(\tau_2)$ for $\tau_1 \geqslant \tau_2$ if $h''_i(y_i) \geqslant 0$ for $i = 1, \ldots, I$.

Proof

Assume, by contradiction, that $S_{i_1}(\tau_1) > S_{i_2}(\tau_2)$ for $i_1 \in M(\tau_1)$, $i_2 \in M(\tau_2)$ and $\tau_1 > \tau_2$.

Equation (8) and Lemma 1 demand that there exists a $\tau_3 \in \langle \tau_2, \tau_1 \rangle$, such that

$$\delta_{i_1} \gamma_{i_1} h'_{i_1}(y_{i_1}(\tau_3)) < \delta_{i_2} \gamma_{i_2} h'_{i_2}(y_{i_2}(\tau_3)) \tag{10}$$

and

$$S_{i_1}(\tau_3) > S_{i_2}(\tau_3)$$

The latter relation in conjunction with eqn. (6) and the continuities of $S_{i_1}(\tau)$ and $S_{i_2}(\tau)$ imply that $\phi_{i_1}(\tau) = 0$ in an interval $I_3 = [\tau_4, \tau_3]$ for $\tau_4 < \tau_3$; it follows from eqn. (2) that $y_{i_1}(\tau)$ is constant for $\tau \in I_3$, while $y_{i_2}(\tau)$ and consequently $h'_{i_2}(y_{i_2}(\tau))$ are not decreasing when $\tau \in I_3$ decreases.

The implication is that eqn. (10) remains valid when τ_3 is replaced by $\tau \in I_3$, which in combination with eqn. (8) imply that $S_{i_1}(\tau) > S_{i_2}(\tau)$ for $\tau \in I_3$. It follows that

$$S_{i_1}(\tau) > S_{i_2}(\tau) \quad \text{for } \tau \in [0, \tau_3) \tag{11}$$

since otherwise there would exist a τ_5 such that $S_{i_1}(\tau_5) = S_{i_2}(\tau_5)$, $S_{i_1}(\tau) > S_{i_2}(\tau)$ for $\tau \in \langle \tau_5, \tau_3 \rangle$, while $\partial S_{i_1}(\tau)/\partial\tau < \partial S_{i_2}(\tau)/\partial\tau$ for $\tau \in \langle \tau_5, \tau_3 \rangle$ by the validity of eqn. (10) with τ_3 replaced by τ, a contradiction. But eqn. (11) demands that $S_{i_1}(0) > S_{i_2}(0)$, violating the assumption $S_{i_1}(0) = S_{i_2}(0) = \beta$, and the lemma follows, q.e.d.

Lemma 2 and eqn. (8) shows that if

$$M(\tau) = \{i_1, \ldots, i_n\}$$

then

$$\delta_{i_1} \gamma_{i_1} h'_{i_1}(y_{i_1}) = \delta_{i_k} \gamma_{i_k} h'_{i_k}(y_{i_k}); \quad k = 1, \ldots, n$$

which, when combined with eqns. (2) and (3), gives the following result:

Theorem 2

Assume that $h''_i(y_i) \geqslant 0$ for $i = 1, \ldots, I$. If $S_i(\tau) < 0$ for $i \in M(\tau)$ an optimal solution is obtained from:

$$\phi_i(\tau) = \begin{cases} 0 \quad \text{for } i \notin M(\tau) \\[2ex] \dfrac{1}{1 + \delta_i^2 \gamma_i h''_i(y_i) \displaystyle\sum_{j \in M(\tau) - \{i\}} 1/\delta_j^2 \gamma_j h''_j(y_j)} & \text{for } i \in M(\tau) \end{cases}$$

and for $i \in \{1, \ldots, I\}$ either $\phi_i(\tau) = 0$ for $\tau \in [0, T]$ or there is a $\tau_0 \in [0, T]$ such that if $\phi_i(\tau_1) > 0$ for a $\tau_1 > \tau_0$ then $\phi_i(\tau) > 0$ for all $\tau \in \langle \tau_0, \tau_1 \rangle$ and $\phi_i(\tau) = 0$ for $\tau \in [0, \tau_0]$.

Proof

If $S_i(\tau)<0$ for $\tau\in[0, T]$ and $i\in M(\tau)$, the value of τ_0 is determined from $S_i(\tau_0)=0$, q.e.d.

The theorem shows that for sufficiently small planning periods the defender allocates all weapons against approaching enemy combat units during the whole planning period. For longer planning periods there is a time τ_0 such that attrition allocations are applied for $\tau>\tau_0$, and directed against supply systems with the largest negative effectiveness value S_i; the weapons being partitioned between the various supply systems according to the values of ϕ_i. When ϕ_i becomes positive, it stays positive until the forward time $t_0=T-\tau_0$, new positive ϕ_i may appear during the time interval $[0, t_0]$. All positive attrition allocations will simultaneously become zero at time t_0 and pure defence allocations are applied in the final time period $[t_0, T]$.

5. Bounds of optimal solutions

Let $\underline{\phi}_i$ and $\bar{\phi}_i$ be lower and upper bounds of the optimal control ϕ_i:

$$\underline{\phi}_i \leqslant \phi_i \leqslant \bar{\phi}_i$$

for instance given by $\underline{\phi}_i(t)=0$ and $\bar{\phi}_i(t)=1$ for $t\in[0, T]$. It will be demonstrated that the following system of differential equations, derived from eqns. (1). (2), (7) and (8), generate bounds of x, y_i, V and S_i, as the notation indicates.

$$\dot{\bar{x}}=\sum_{i=1}^{I}\gamma_i h_i(\underline{y}_i)-\beta g(\bar{x})\left(1-\sum_{i=1}^{I}\bar{\phi}_i\right)$$

$$\dot{\underline{x}}=\sum_{i=1}^{I}\gamma_i h_i(\bar{y}_i)-\beta g(\underline{x})\left(1-\sum_{i=1}^{I}\underline{\phi}_i\right)$$

$$\dot{\bar{y}}_i=\delta_i g(\underline{x})\bar{\phi}_i; \quad i=1, \ldots, I$$

$$\dot{\underline{y}}_i=\delta_i g(\bar{x})\underline{\phi}_i; \quad i=1, \ldots, I$$

$$\frac{\partial \bar{V}}{\partial \tau}=-(\bar{V}+1)\beta g'(\underline{x})+g'(\underline{x})\sum_{i=1}^{I}\bar{\phi}_i \min(0, \underline{S}_i)$$

$$\frac{\partial \underline{V}}{\partial \tau}=-(\underline{V}+1)\beta g'(\bar{x})+g'(\bar{x})\sum_{i=1}^{I}\underline{\phi}_i \min(0, \bar{S}_i)$$

$$\frac{\partial \bar{S}_i}{\partial \tau}=\beta\frac{\partial \bar{V}}{\partial \tau}+\delta_i\gamma_i(\underline{V}+1)h'_i(\bar{y}_i); \quad i=1, \ldots, I$$

$$\frac{\partial \underline{S}_i}{\partial \tau}=\beta\frac{\partial \underline{V}}{\partial \tau}+\delta_i\gamma_i(\bar{V}+1)h'_i(\underline{y}_i); \quad i=1, \ldots, I$$

with the initial conditions:

$$\underline{x}(0)=\bar{x}(0)=0$$

$$\underline{y}_i(0)=\bar{y}_i(0)=0; \quad i=1, \ldots, I$$

$$\underline{V}(0)=\bar{V}(0)=0$$

$$\underline{S}_i(0)=\bar{S}_i(0)=\beta; \quad i=1, \ldots, I$$

Theorem 3

If $g''(x)\geqslant 0$ and $h''_i(y_i)\geqslant 0$ and if x, V and y_i, S_i for $i=1, \ldots, I$ correspond to an optimal solution of P, it follows that:

$$\underline{x}\leqslant x\leqslant \bar{x}, \quad \underline{V}\leqslant V\leqslant \overline{V} \quad \text{and} \quad \underline{y}_i\leqslant y_i\leqslant \bar{y}_i, \quad \underline{S}_i\leqslant S_i\leqslant \bar{S}_i \quad \text{for } i=1, \ldots, I$$

Proof

Extend the definitions of the functions g and h_i by:

$$g(x)=g(0)+g'(0)x \quad \text{for } x\leqslant 0$$

$$h_i(y_i)=h_i(0)+h'_i(0)y_i \quad \text{for } y_i\leqslant 0$$

and consider the differential equations for $\underline{x}$ and $\bar{y}_i$ with the initial conditions:

$$\underline{x}(0)=\eta<0 \tag{12}$$

$$\bar{y}_i(0)=\xi_i>0 \tag{13}$$

It follows that

$$\underline{x}(0)<x(0)$$

$$\bar{y}_i(0)>y_i(0)$$

and consequently that $\underline{x}(t)<x(t)$ and $\bar{y}_i(t)>y_i(t)$ for t sufficiently close to 0. Assume, by contradiction, that there exists a t such that $\underline{x}(t)=x(t)$ or $\bar{y}_i(t)=y_i(t)$ and select the smallest such value of t, denoted by t_0. If $\underline{x}(t_0)=x(t_0)$, it follows from $\underline{x}(t)<x(t)$ and $\bar{y}_i(t)>y_i(t)$ for $t\in[0, t_0]$ that $\dot{\underline{x}}(t)\leqslant \dot{x}(t)$ for $t\in[0, t_0)$, which is impossible in combination with $\underline{x}(0)=\eta<x(0)=0$ and $\underline{x}(t_0)=x(t_0)$. A similar argument applies if $\bar{y}_i(t_0)=y_i(t_0)$. The implication is that $\underline{x}(t)<x(t)$ and $\bar{y}_i(t)>y_i(t)$ for the initial conditions (12) and (13); the result for $\eta=\xi_i=0$ follows from the continuity of solutions of differential equations with respect to initial conditions, see Coddington and Levinson (1955). The other statements of the theorem follow from the assumptions that $g''(x)\geqslant 0$ and $h''_i(y_i)\geqslant 0$, q.e.d.

It may be possible to apply the values of $\underline{S}_i$ and $\bar{S}_i$ to improve the bounds $\underline{\phi}_i$ and $\bar{\phi}_i$; for instance: if $\underline{S}_i(\tau_i)=0$ and $\underline{S}_i(\tau)>0$ for $\tau\in[0, \tau_i]$, it follows that $\bar{\phi}_i(\tau)=0$ for $\tau\in[0, \tau_i]$. The improved $\underline{\phi}_i$ and $\bar{\phi}_i$ are then used to derive new bounds $\underline{x}$, $\bar{x}$, $\underline{y}_i$, $\bar{y}_i$, $\underline{V}$, $\overline{V}$, $\underline{S}_i$ and $\bar{S}_i$.

In the application of numerical solution techniques to the problem P, the bounds of x, y_i, V and S_i given in theorem 3, are useful in the derivation of initial estimates of these quantities.

6. Pure defence criteria

An optimal solution (ϕ_i) of P is defined to be a pure defence strategy if and only if $\phi_i(t)=0$ for $t\in[0, T]$ and $i=1, \ldots, I$.

Theorem 4

If $g''(x)\geqslant 0$ and $h_i''(y_i)\geqslant 0$ for $i=1, \ldots, I$ a pure defence strategy is applied if

$$\min_i \min_{\tau\in[0, T]} \underline{S}_i(\tau)>0$$

or if

$$\min\left[-\beta^2 g'\left(\sum_{i=1}^{I} \gamma_i h_i(0)T\right)+\delta_i\gamma_i h_i(0)\right]\geqslant 0$$

Proof

The first criterion follows from the definition of $\underline{S}_i(\tau)$.

In the demonstration of the second criterion note that $S_i(0)=\beta$ for $i=1, \dots, I$, and the continuity of $S_i(\tau)$, imply that $\phi_i(\tau)=0$ for $i=1, \dots, I$ and $\tau\in[0, \tau_1]$ for some $\tau_1>0$. If $\phi_i=0$ for $i=1, \dots, I$, eqns. (7) and (8) demand that:

$$\frac{\partial S_i}{\partial\tau}=(V+1)[-\beta^2 g'(x)+\delta_i\gamma_i h'_i(y_i)] \quad i=1, \dots, I$$

which in combination with

$$g''(x)\geqslant 0, \quad h''_i(y_i)\geqslant 0 \quad \text{for } i=1, \dots, I$$

and

$$0\leqslant x\leqslant \sum_{i=1}^{I} \gamma_i h_i(0)T$$

$$0\leqslant y_i\leqslant \delta_i g_i(0)T$$

demands that:

$$\frac{\partial S_i}{\partial\tau}\geqslant 0; \quad i=1, \dots, I$$

Since $S_i>0$ for $i=1, \dots, I$, the optimality condition (6) requires that $\phi_i=0$ for all $t\in[0, T]$ and $i=1, \dots, I$, q.e.d.

7. Concluding remarks

Assuming that

$$g(x)=g(0)-Gx$$

$$h_i(y_i)=h_i(0)-H_i y_i; \quad i=1, \dots, I$$

and that $\phi_i=0$ for $i=1, \dots, I$, it follows that

$$\frac{\partial S_i}{\partial\tau}=\exp(\beta G\tau)[\beta^2 G-\delta_i\gamma_i H_i]; \quad i=1, \dots, I$$

Assume, without loss of generality, that:

$$\delta_1\gamma_1 H_1=\max_i(\delta_i\gamma_i H_i)$$

A pure defence strategy applies if $[\beta^2 G-\delta_1\gamma_1 H_1]\geqslant 0$; for G, H_i and γ_i fixed this gives a parabola

$$\delta_i=\frac{G}{\gamma_i H_i}\beta^2$$

in the (β, δ_i) plane, separating the pure defence strategies from the possibly mixed attrition strategies; a family of parabolas may be parameterized by the value of $(G/\gamma_1 H_1)$.

Outside the pure defence region, solution of eqn. (14) gives:

$$S_1(\tau)=\beta+[\beta^2 G-\delta_1\gamma_1 H_1]\frac{\exp(\beta G\tau)-1}{\beta G}$$

and

$$S_1(\tau)\leqslant S_i(\tau); \quad i=1, \ldots, I$$

demanding that $\phi_i(t)=0$ for $t\in[0, T]$ and $i=2, \ldots, I$. Since $S_1(\tau_1)=0$ for

$$\tau_1=\frac{1}{\beta G}\ln\left(\frac{\delta_1\gamma_1 H_1}{\delta_1\gamma_1 H_1-\beta^2 G}\right)$$

is seen that

$$\phi_1(t)=0 \quad \text{for } t\in[0, T] \quad \text{if } \tau_1\geqslant T$$

$$\phi_1(t)=\left\{\begin{matrix} 1 & \text{for } t\in[0, T-\tau_1] \\ & \\ 0 & \text{for } t\in[T-\tau_1, T] \end{matrix}\right\} \quad \text{if } \tau_1<T$$

Considering the general problem P, the result of Theorem 2 can be used to construct a numerical solution algorithm based on a search for a number τ_0, such that all positive allocations ϕ_i, if any, become zero for $\tau=\tau_0$. Specifically, order the enemy targets $i\in\{1, \ldots, I\}$ such that

$$\delta_1\gamma_1 h'_1(0)\leqslant\delta_2\gamma_2 h'_2(0)\leqslant\ldots\leqslant\delta_I\gamma_I h'_I(0)$$

and integrate eqns. (1) and (2) in the forward direction with the initial conditions (4) and (5) and $\phi_1(\tau)$ given by Theorem 2, and the set $M(\tau)$ determined from $M(\tau)=\{1, \ldots, k\}$ iff for $i=1, \ldots, k$:

$$\delta_k\gamma_k h'_k(0)\leqslant\delta_i\gamma_i h'_i(y_i)<\delta_{k+1}\gamma_{k+1}h'_{k+1}(0)$$

The integration proceeds until a selected time $t_0=t-\tau_0$, assuming that $\phi_i(\tau)=0$ for $\tau\in[0, \tau_0]$ and $i=1, \ldots, I$. With $y_i(t)=y_i(t_0)$ for $t\in[t_0, T]$ and $x(t)$ given from the solution of the differential equation:

$$\dot{x}=\sum_{i=1}^{I}\gamma_i h_i(y_i(t_0))-\beta g(x); \quad t\in[t_0, T]$$

eqns. (7) and (8) for V and S_i for $i=1, \ldots, I$ are solved retrogressively, noting that Theorem 2 implies that $S_{i_1}(\tau)=S_{i_2}(\tau)$ for any $\{i_1, i_2\}\subset M(\tau_0)$. A value τ_0^* is determined from $S_{i_1}(\tau_0^*)=0$ for $i_1\in M(\tau_0)$. An optimal solution has been found if $\tau_0=\tau_0^*$ (with sufficient accuracy); otherwise a trial and error procedure is started, in which τ_0 is increased or decreased depending on whether $\tau_0<\tau_0^*$ or $\tau_0>\tau_0^*$.

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