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A correction of a common error in truncated second-order non-linear filters

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A rederivation of the truncated second-order non-linear filter reveals that a significant error appears in previous derivations of this filter. What has previously been termed the modified truncated second-order filter will be shown to be, provided a small correction is made in the discrete-time case, the correct form of the truncated second-order filter.

1. Introduction

Since the introduction of the Kalman filter (Kalman (1960)), similar techniques have been developed for non-linear dynamical systems. Most of these works were carried out and published in the late sixties, and a thorough presentation of the different techniques was made in the book by Jazwinski (1970).

Of all the non-linear filters which have appeared in the literature, especially the so-called second-order filters, i.e., the *truncated second-order filter* and the *Gaussian second-order filter* seem to have attracted much attention. Non-linearities are in both these filters carried to second order only, but while third and higher order central moments are neglected in the truncated second-order filter, fourth-order moments are taken into account in the Gaussian second-order filter by approximating the conditional pdf. which is involved by a normal or Gaussian pdf. Simulation or experimental results with these filters clearly show that they may improve the estimates compared with the extended Kalman filter, but the improvement may depend on the system non-linearities and the magnitude of the plant and measurement noises, see Schwartz and Stear (1968), Jazwinski (1966 b), Carney and Goldwyn (1967), and Henriksen and Olsen (1977).

Andrade Netto *et al.* (1976) have compared several non-linear filters for discrete-time systems and concluded that the truncated second-order filter should not be used because of a term in the covariance equation which tends to decrease the covariance matrix. Jazwinski (1970) also makes some comments on the apparent discrepancy between the truncated and the Gaussian second-order filters where a term enters the covariance equations of the two filters with opposite signs. Obviously disquieted by this, but with no apparent theoretical justification, Jazwinski suggests a third type of second-order filter, the so-called *modified second-order filter*, where the aforementioned term has been removed.

The truncated second-order non-linear filter was first developed by Jazwinski (1966 a) and independently by Bass *et al.* (1966). What we are about to show in this paper is that the truncated second-order filter, as it appears in the literature, actually is wrong since a very important implication of the assumptions on which it is based

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has been overlooked. In fact, it will be shown that the modified second-order filter is the correct form of what has been termed the truncated second-order filter.

The paper is organized as follows. In § 2 we prove the Proposition which is used to correct the truncated second-order filter, and derive the correct form of this filter for discrete-time systems. The correct form of the truncated second-order filter for continuous-time systems is then derived in § 3.

2. Discrete-time truncated second-order filter

Consider a discrete-time non-linear system described by the equations

$$x_{k+1} = f(x_k, t_k) + G(x_k, t_k)v_k \tag{1}$$

$$y_k = h(x_k, t_k) + w_k \tag{2}$$

where x_k and y_k are the state and observation vectors, respectively. The noise processes $\{v_k\}$ and $\{w_k\}$ are assumed to be mutually independent zero-mean white processes, both assumed to be independent of x_0. The covariance matrices of v_k and w_k are denoted by, respectively, V_k and W_k.

Define Y_k to be the sequence of observations $y_0, y_1, \ldots, y_k$ up to time k, viz.

$$Y_k = (y_0, y_1, \ldots, y_k),$$

and define $\hat{x}_{i|k}$ to be the expectation of x_i with respect to $\sigma(Y_k)$, the σ-algebra generated by Y_k, viz.

$$\hat{x}_{i|k} = E(x_i \,|\, \sigma(Y_k)) \triangleq E(x_i \,|\, Y_k) \tag{3}$$

Similarly, define

$$X_{i|k} = E((x_i - \hat{x}_{i|k})(x_i - \hat{x}_{i|k})^{\mathrm{T}} \,|\, \sigma(Y_k)) \triangleq E((x_i - \hat{x}_{i|k})(x_i - \hat{x}_{i|k})^{\mathrm{T}} \,|\, Y_k) \tag{4}$$

We also introduce the notation

$$E_k(\,.\,) \triangleq E(\,.\,|\sigma(Y_k)) \tag{5}$$

The exact equations for $\hat{x}_{k+1|k}$ and $X_{k+1|k}$ are from eqns. (1)–(2) found to be

$$\hat{x}_{k+1|k} = E_k(f(x_k, t_k)) \tag{6}$$

$$X_{k+1|k} = E_k(f(x_k, t_k)f^{\mathrm{T}}(x_k, t_k)) + E_k(G(x_k, t_k)V_kG^{\mathrm{T}}(x_k, t_k)) - \hat{x}_{k+1|k}\hat{x}_{k+1|k}^{\mathrm{T}} \tag{7}$$

Now, let us assume third and higher order central moments to be negligible. This is appropriate if the conditional pdf. is almost symmetrical and concentrated near its mean. Carrying the non-linearities to second order only and neglecting third and higher order central moments, we find

$$\hat{x}_{k+1|k} = f(\hat{x}_{k|k}, t_k) + \tfrac{1}{2}(X_{k|k}f_{xx}(\hat{x}_{k|k}, t_k)) \tag{8}$$

with the ith component of the vector $(X_{k|k}f_{xx}(\hat{x}_{k|k}, t_k))$ given by

$$(X_{k|k}f_{xx}(\hat{x}_{k|k}, t_k))_i = \sum_{j,\,l=1}^{n} X_{k|k}{}^{jl} \frac{\partial^2 f_i}{\partial x_k{}^j \, \partial x_k{}^l}(\hat{x}_{k|k}, t_k) \tag{9}$$

where $X_{k|k}{}^{jl}$ is element (j, l) of $X_{k|k}$, f_i is the ith component of f while $x_k{}^j$ is the jth component of x_k.

Before we proceed, let us make a note about the magnitude of fourth-order central moments compared to the square of second-order central moments. If ξ is a normal random variable with variance σ^2, we know that the fourth-order central moment of ξ is $3\sigma^4$ which certainly is greater than σ^4. A more general result is provided by the following:

Proposition

Let ξ be a random zero-mean vector, and let $F(\xi\xi^T)$ be a vector function of $\xi\xi^T$. Then†

$$E(F(\xi\xi^T)F^T(\xi\xi^T)) \geq E(F(\xi\xi^T))E(F^T(\xi\xi^T)) \tag{10}$$

Proof. Absolutely trivial. We have

$$\begin{aligned} &E(F(\xi\xi^T)F^T(\xi\xi^T)) - E(F(\xi\xi^T))E(F^T(\xi\xi^T)) \\ &\qquad = E\{[F(\xi\xi^T) - E(F(\xi\xi^T))][F(\xi\xi^T) - E(F(\xi\xi^T))]^T\} \\ &\qquad \geq 0 \end{aligned}$$

□

Carrying the non-linearities in eqn. (7) to second order about $\hat{x}_{k|k}$, taking expectations with respect to $\sigma(Y_k)$, and neglecting third-order central moments (while retaining for the moment fourth-order central moments), we find

$$\begin{aligned} X_{k+1|k} = {} & f_x(\hat{x}_{k|k}, t_k) X_{k|k} f_x^T(\hat{x}_{k|k}, t_k) \\ & + \tfrac{1}{4} E_k\{[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T f_{xx}(\hat{x}_{k|k}, t_k)][(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T f_{xx}(\hat{x}_{k|k}, t_k)]^T\} \\ & + G(\hat{x}_{k|k}, t_k) V_k G(\hat{x}_{k|k}, t_k)^T + (X_{k|k} V_k G_x^2(\hat{x}_{k|k}, t_k)) \\ & + \tfrac{1}{2}(X_{k|k} V_k G_{xx}(\hat{x}_{k|k}, t_k) G(\hat{x}_{k|k}, t_k)) + \tfrac{1}{2}(X_{k|k} V_k G_{xx}(\hat{x}_{k|k}, t_k) G(\hat{x}_{k|k}, t_k))^T \\ & - \tfrac{1}{4}(X_{k|k} f_{xx}(\hat{x}_{k|k}, t_k))(X_{k|k} f_{xx}(\hat{x}_{k|k}, t_k))^T \end{aligned} \tag{11}$$

where the components of the vector $[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T f_{xx}(\hat{x}_{k|k}, t_k)]$, and the elements of the matrices $(X_{k|k} V_k G_x^2(\hat{x}_{k|k}, t_k))$ and $(X_{k|k} V_k G_{xx}(\hat{x}_{k|k}, t_k) G(\hat{x}_{k|k}, t_k))$ are given by, respectively

$$[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T f_{xx}(\hat{x}_{k|k}, t_k)]_i = \sum_{j,l=1}^{n} (x_k^j - \hat{x}_{k|k}^j)(x_k^l - \hat{x}_{k|k}^l) \frac{\partial^2 f_i}{\partial x_k^j \, \partial x_k^l} (\hat{x}_{k|k}, t_k) \tag{12}$$

$$(X_{k|k} V_k G_x^2(\hat{x}_{k|k}, t_k))_{iq} = \sum_{j,l=1}^{s} \sum_{p,t=1}^{n} X_{k|k}^{pt} V_k^{jl} \frac{\partial G_{ij}}{\partial x_k^p} (\hat{x}_{k|k}, t_k) \frac{\partial G_{ql}}{\partial x_k^t} (\hat{x}_{k|k}, t_k) \tag{13}$$

$$(X_{k|k} V_k G_{xx}(\hat{x}_{k|k}, t_k) G(\hat{x}_{k|k}, t_k))_{iq} = \sum_{j,l=1}^{s} \sum_{p,t=1}^{n} X_{k|k}^{pt} V_k^{jl} \frac{\partial^2 G_{ij}}{\partial x_k^p \, \partial x_k^t} (\hat{x}_{k|k}, t_k) G_{ql}(\hat{x}_{k|k}, t_k) \tag{14}$$

We have assumed x_k and v_k to be n and s vectors, respectively. Elements of the matrices $X_{k|k}$, V_k, and G are denoted by, respectively, $X_{k|k}^{pt}$, V_k^{jl}, and G_{ij}.

† For two symmetric matrices A and B, $A \geq B$ or $A - B \geq 0$ means that $A - B$ is positive semidefinite.

Since we have assumed fourth-order central moments to be negligible, the second right-hand term in eqn. (11) can be dropped. The remainder of that equation then constitutes the covariance equation for the one-stage predicted estimate as it appears in, say, Jazwinski (1970). However, by the previous proposition, the order of magnitude of the second right-hand term in eqn. (11) is at least as great as the last term. So if we assume that the second right-hand term is negligible, it would be not only illogical but even wrong to retain the last term since the sum of these two terms is non-negative definite. Neglecting these two terms in eqn. (11) yields

$$X_{k+1|k}=f_x(\hat{x}_{k|k}, t_k)X_{k|k}f_x^{\mathrm{T}}(\hat{x}_{k|k}, t_k)+G(\hat{x}_{k|k}, t_k)V_kG^{\mathrm{T}}(\hat{x}_{k|k}, t_k)$$
$$+(X_{k|k}V_kG_x^{\,2}(\hat{x}_{k|k}, t_k))+\tfrac{1}{2}(X_{k|k}V_kG_{xx}(\hat{x}_{k|k}, t_k)G(\hat{x}_{k|k}, t_k))$$
$$+\tfrac{1}{2}(X_{k|k}V_kG_{xx}(\hat{x}_{k|k}, t_k)G(\hat{x}_{k|k}, t_k))^{\mathrm{T}} \qquad (15)$$

Andrade Netto *et al.* (1976) conclude that the truncated filter should not be used because of the last term in eqn. (11) which has a tendency to decrease the covariance matrix and eventually may force some of the diagonal terms to be negative. Since we now know that this term is not properly present and that eqn. (15) is the correct covariance equation, their conclusion can therefore be discarded.

In order to derive the equations for the filtered estimates $\hat{x}_{k+1|k+1}$ and $X_{k+1|k+1}$, the following form usually appears in the literature

$$\hat{x}_{k+1|k+1}=\hat{x}_{k+1|k}+A(y_{k+1}-\hat{y}_{k+1|k}) \qquad (16)$$

$$X_{k+1|k+1}=B+C(y_{k+1}-\hat{y}_{k+1|k}) \qquad (17)$$

Furthermore, if $C=0$ is assumed, the form is usually termed the *modified* truncated second-order filter.

Filtering equations of the form given by eqns. (16)–(17) have been derived both for the truncated and Gaussian second-order filters, ending up with similar expressions for the tensor C in the two cases but with opposite signs. Obviously disquieted by this, Jazwinski (1970) also suggests forms with $C=0$, the so-called modified second-order filters.

The elaborate derivation of expressions for the matrices A and B, and the tensor C can be found in Jazwinski (1970). A careful computation, applying the previous proposition, actually reveals that $C=0$ (see also the next section where the result $C=0$ is shown to hold in the continuous-time case). Therefore, the term modified is quite superfluous and should not be used in connection with the truncated second-order filter. Assuming that third and fourth order central moments are negligible will in fact imply that the truncated second-order filter becomes "modified" if we still may use that term.

The filtering equations are, once $C=0$ has been verified, quite easily derived by assuming Gaussian distributions and simply computing the first and second order moments of the posterior pdf. $p(x_{k+1} \mid Y_{k+1})$ by carrying non-linearities in h to second order, see eqns. (20)–(22).

Summing up, the true and only truncated second-order filter is of the following form:

Prediction

$$\hat{x}_{k+1|k+1}=f(\hat{x}_{k|k}, t_k)+\tfrac{1}{2}(X_{k|k}f_{xx}(\hat{x}_{k|k}, t_k)) \qquad (18)$$

$$X_{k+1|k}=f_x(\hat{x}_{k|k}, t_k)X_{k|k}f_x^{\mathrm{T}}(\hat{x}_{k|k}, t_k)+G(\hat{x}_{k|k}, t_k)V_kG^{\mathrm{T}}(\hat{x}_{k|k}, t_k)$$
$$+(X_{k|k}V_kG_x^2(\hat{x}_{k|k}, t_k))+\tfrac{1}{2}(X_{k|k}V_kG_{xx}(\hat{x}_{k|k}, t_k)G(\hat{x}_{k|k}, t_k))$$
$$+\tfrac{1}{2}(X_{k|k}V_kG_{xx}(\hat{x}_{k|k}, t_k)G(\hat{x}_{k|k}, t_k))^{\mathrm{T}} \quad (19)$$

Filtering

$$\hat{x}_{k+1|k+1}=\hat{x}_{k+1|k}+K_{k+1}[y_{k+1}-h(\hat{x}_{k+1|k}, t_{k+1})-\tfrac{1}{2}(X_{k+1|k}h_{xx}(\hat{x}_{k+1|k}, t_{k+1}))] \quad (20)$$

$$X_{k+1|k+1}=[I-K_{k+1}h_x(\hat{x}_{k+1|k}, t_{k+1})]X_{k+1|k} \quad (21)$$

where

$$K_{k+1}=X_{k+1|k}h_x^{\mathrm{T}}(\hat{x}_{k+1|k}, t_{k+1})$$
$$\times [h_x(\hat{x}_{k+1|k}, t_{k+1})X_{k+1|k}h_x^{\mathrm{T}}(\hat{x}_{k+1|k}, t_{k+1})+W_{k+1}]^{-1} \quad (22)$$

The ith component of the vector $(X_{k+1|k}h_{xx}(\hat{x}_{k+1|k}, t_{k+1}))$ in eqn. (20) is of the form

$$(X_{k+1|k}h_{xx}(\hat{x}_{k+1|k}, t_{k+1}))_i=\sum_{j,l=1}^{m} X_{k+1|k}{}^{jl}\frac{\partial^2 h_i}{\partial x_{k+1}{}^j\,\partial x_{k+1}{}^l}(\hat{x}_{k+1|k}, t_{k+1}) \quad (23)$$

The terms which erroneously have been retained in previous papers on truncated second-order filters will appear in eqns. (19) and (22). The filter above is almost identical to what has been termed the modified truncated second-order filter in Jazwinski (1970). However, another term,

$$-\tfrac{1}{4}(X_{k+1|k}h_{xx}(\hat{x}_{k+1|k}, t_{k+1}))(X_{k+1|k}h_{xx}(\hat{x}_{k+1|k}, t_{k+1}))^{\mathrm{T}}$$

appears inside the brackets in eqn. (22) in that book. Using the previous proposition we also find that this term should be dropped.

3. Continuous-time truncated second-order filter

Consider a system governed by the following set of Itô stochastic differential equations:

$$dx(t)=f(x(t), t)\,dt+G(x(t), t)\,d\beta(t) \quad (25)$$

$$dz(t)=h(x(t), t)\,dt+d\eta(t) \quad (26)$$

where $\{\beta(t), t\geq t_0\}$ and $\{\eta(t), t\geq t_0\}$ are mutually independent Wiener processes, both assumed to be independent of $x(t_0)$. Furthermore, assume $E(d\beta(t)\,d\beta^{\mathrm{T}}(t))=V(t)\,dt$ and $E(d\eta(t)\,d\eta^{\mathrm{T}}(t))=W(t)\,dt$.

Define $\hat{x}(t|t)$ and $X(t|t)$ to be, respectively, the mean and covariance matrix of $x(t)$ given $Z(t)=\{z(\tau), t_0\leq\tau\leq t\}$. Furthermore, let E_t denote the expectation operator with respect to $\sigma(Z(t))$. Equations for the evolution of $\hat{x}(t|t)$ and $X(t|t)$ are derived in Jazwinski (1970), pp. 182–184. For the conditional mean $\hat{x}(t|t)$ we have

$$d\hat{x}(t|t)=E_t(f(x(t), t))\,dt+[E_t(x(t)h^{\mathrm{T}}(x(t), t))$$
$$-\hat{x}(t|t)E_t(h^{\mathrm{T}}(x(t), t))]W^{-1}(t)[dz(t)-E_t(h(x(t), t))\,dt] \quad (27)$$

Proceeding in a similar way as in the discrete-time case, we end up with

$$d\hat{x}(t|t)=[f(\hat{x}(t|t),t)+\tfrac{1}{2}(X(t|t)f_{xx}(\hat{x}(t|t),t))]\,dt$$
$$+K(t)\{dz(t)-[h(\hat{x}(t|t),t)+\tfrac{1}{2}(X(t|t)h_{xx}(\hat{x}(t|t),t))]\,dt\} \quad (28)$$

where

$$K(t)=X(t|t)h_x^{\mathrm{T}}(\hat{x}(t|t),t)W^{-1}(t) \quad (29)$$

$$(X(t|t)f_{xx}(\hat{x}(t|t),t))_i=\sum_{i,k=1}^{n}X_{jk}(t|t)\frac{\partial^2 f_i}{\partial x_j\partial x_k}(\hat{x}(t|t),t) \quad (30)$$

$$(X(t|t)h_{xx}(\hat{x}(t|t),t))_i=\sum_{j,k=1}^{n}X_{jk}(t|t)\frac{\partial^2 h_i}{\partial x_j\partial x_k}(\hat{x}(t|t),t) \quad (31)$$

The (i,j)th component of $X(t|t)$ will satisfy the equation

$$dX_{ij}=\{[E_t(x_if_j)-\hat{x}_iE_t(f_j)]+[E_t(x_jf_i)-\hat{x}_jE_t(f_i)]+E_t(GVG^{\mathrm{T}})_{ij}$$
$$-[E_t(x_ih)-\hat{x}_iE_t(h)]^{\mathrm{T}}W^{-1}[E_t(x_jh)-\hat{x}_jE_t(h)]\}\,dt$$
$$+[E_t(x_ix_jh)-E_t(x_ix_j)E_t(h)-\hat{x}_iE_t(x_jh)$$
$$-\hat{x}_jE_t(x_ih)+2\hat{x}_i\hat{x}_jE_t(h)]^{\mathrm{T}}W^{-1}[dz-E_t(h)dt] \quad (32)$$

where all arguments of the functions have been dropped for the sake of simplicity.

First, consider the term in front of $[dz-E_t(h)\,dt]$. Taking the non-linearities to second order, we find

$$[E_t(x_ix_jh)-E_t(x_ix_j)E_t(h)-\hat{x}_iE_t(x_jh)-\hat{x}_jE_t(x_ih)+2\hat{x}_i\hat{x}_jE_t(h)]$$
$$=\tfrac{1}{2}E_t\{(x_i-\hat{x}_i)(x_j-\hat{x}_j)[(x-\hat{x})(x-\hat{x})^{\mathrm{T}}h_{xx}]\}-\tfrac{1}{2}X_{ij}(Xh_{xx}) \quad (33)$$

where the term $[(x-\hat{x})(x-\hat{x})^{\mathrm{T}}h_{xx}]$ has a similar interpretation as the term (Xh_{xx}) which is defined in eqn. (31). Jazwinski (1966 a, 1970) again neglects the fourth-order central moments in eqn. (33), whereas the term $-\tfrac{1}{2}X_{ij}(Xh_{xx})$ is being retained, thus obtaining a random forcing term in the covariance equation. However, the proper approximation of the right-hand side of eqn. (33) is actually 0 which completely eliminates the random forcing term from the covariance equation.

Proceeding as previously, we finally end up with the covariance equation being of the form

$$\frac{d}{dt}X(t|t)=f_x(\hat{x}(t|t),t)X(t|t)+X(t|t)f_x^{\mathrm{T}}(\hat{x}(t|t),t)$$
$$+G(\hat{x}(t|t),t)V(t)G^{\mathrm{T}}(\hat{x}(t|t),t)$$
$$+(X(t|t)V(t)G_x^{2}(\hat{x}(t|t),t))$$
$$+\tfrac{1}{2}(X(t|t)V(t)G_{xx}(\hat{x}(t|t),t)G(\hat{x}(t|t),t))$$
$$+\tfrac{1}{2}(X(t|t)V(t)G_{xx}(\hat{x}(t|t),t)G(\hat{x}(t|t),t))^{\mathrm{T}}$$
$$-X(t|t)h_x^{\mathrm{T}}(\hat{x}(t|t),t)W^{-1}(t)h_x(\hat{x}(t|t),t)X(t|t) \quad (34)$$

where the matrices $(X(t|t)V(t)G_x^{2}(\hat{x}(t|t),t))$ and $(X(t|t)V(t)G_{xx}(\hat{x}(t|t),t)G(\hat{x}(t|t),t))$ have a similar interpretation as in the previous section.

Summing up, the correct form of the truncated second-order non-linear filter for continuous-time systems consists of the following equations:

$$d\hat{x}(t|t)=[f(\hat{x}(t|t),t)+\tfrac{1}{2}(X(t|t)f_{xx}(\hat{x}(t|t),t))]\,dt \\ +K(t)\{dz(t)-[h(\hat{x}(t|t),t)+\tfrac{1}{2}(X(t|t)h_{xx}(\hat{x}(t|t),t))]\,dt\} \quad (35)$$

$$\frac{d}{dt}X(t|t)=f_x(\hat{x}(t|t),t)X(t|t)+X(t|t)f_x^{\mathrm{T}}(\hat{x}(t|t),t) \\ +G(\hat{x}(t|t),t)V(t)G^{\mathrm{T}}(\hat{x}(t|t),t)+(X(t|t)V(t)G_x^{2}(\hat{x}(t|t),t)) \\ +\tfrac{1}{2}(X(t|t)V(t)G_{xx}(\hat{x}(t|t),t)G(\hat{x}(t|t),t)) \\ +\tfrac{1}{2}(X(t|t)V(t)G_{xx}(\hat{x}(t|t),t)G(\hat{x}(t|t),t))^{\mathrm{T}}-K(t)W(t)K^{\mathrm{T}}(t) \quad (36)$$

where

$$K(t)=X(t|t)h_x^{\mathrm{T}}(\hat{x}(t|t),t)W^{-1}(t) \quad (37)$$

The latter filter is identical to what has previously been called the modified second-order filter for continuous-time systems. However, as we have pointed out previously, there is no need to use the term modified.

4. Conclusion

We have rederived the truncated second-order non-linear filter for both discrete-time and continuous-time systems and have shown that previous derivations contain a significant error. This is due to the fact that an important implication of the assumptions has been overlooked in previous papers. The derivations in this paper reveal that what has previously been termed the modified truncated second-order filter is the correct form of the truncated filter provided a small correction is made in the discrete-time case. The term modified can therefore be dropped in connection with the truncated second-order filter.

A truncated second-order filter for discrete-time implicit systems has been derived by Henriksen (1979). However, in order to obtain a computationally feasible solution, a stronger set of assumptions than appears in this paper had to be made.

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