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Nuclear reaction control by multistage mathematical programming

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The method of Multistage Mathematical Programming (MMP) has been adapted for optimal control of the spatial power distribution in nuclear reactors. Changes in power distribution arise from variations in the operational conditions of the reactor, and excite oscillations which in large reactors may approach the stability limit.

The dynamic process giving rise to the oscillations is connected to the production, absorption and decay of xenon and iodine isotopes which are produced during fission of uranium. Using MMP, efficient use of the controllers can be precalculated to damp these oscillations. Simulations of MMP control on computer models of large reactors have demonstrated the power of the method. The ability of MMP to take into account hard constraints on controllers and state variables is considered an especially important feature.

1. Introduction

To improve the operational flexibility and capability of large light water reactors, and to minimize the fuel stresses, increasing attention is being paid to strategies for controlling the spatial power distribution in the reactor core. Oscillations are easily excited if the total power is changed. Daily load cycling with full power operation during the day, and reduced power during the night is becoming more and more desirable as nuclear reactors deliver an increasingly large fraction of the electricity production to the grid.

Two classes of power distribution control methods have been studied at the Halden Project. One is the use of feedback methods for control of the power distribution toward some specified setpoint. Formulation of the feedback control problem has been made both in the time and the frequency domain, using the State Variable Feedback and Multivariable Frequency Response methods respectively.

The other activity in the control field has been aimed at developing a method for optimization of the core behaviour for a period of time ahead, typically 5 to 10 hours. The Multistage Mathematical Programming (MMP) method has proved to be useful for this purpose. Especially useful for the application in nuclear reactor power distribution control, is the ability of MMP to take into account hard constraints on controllers and state variables. This paper will describe the control problem, the Multistage Mathematical Programming method as applied in solving the problem, and results from simulations where MMP has been used.

2. The control problem

The description here applies mainly to the most common reactor type, the pressurized light water reactor (PWR), where the spatial effects are fairly strong and which has been modelled for the simulations.

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2.1. *Fundamental PWR core characteristics*

The energy is released in neutron induced fissions, in which the fissionable atoms split into fission products i.e. other atoms. The released energy heats up the fuel pins, which are cooled by water under high pressure. The water also acts as neutron moderator, slowing down the speed of the neutrons born in the fission to speeds where they are more effective in causing new fissions. The migration and absorption of neutrons depends on the atomic composition and temperatures in the core, which change continuously during power operation of the reactor.

The fuel pins (4 m high, 1 cm diameter) are assembled together to form fuel elements. There are about 200 pins in each element and about 200 elements in the whole core. The coolant flows along the fuel pins from the bottom of the core upwards. In certain fuel elements some of the fuel pins are replaced by tubes, in which strongly neutron-absorbing material can be entered from above into the core. These absorber pins from control rods, which are moved as syncronized control rod banks such that all rods in one bank are equally deeply inserted in the core. The control rods reduce the power generation in their surroundings and can thus be used for changing both the total power and the power distribution in the core. Another type of power controller is the soluble boric acid (a strong neutron absorber) concentration in the coolant. It has an almost homogeneous distribution and uniform effect on the power. The control rod banks can be operated quickly, but the changing of the soluble boron concentration is fairly slow.

The reactor is normally equipped with neutron detectors in and outside the core, and thermocouples in the core. The power distribution in the core can be inferred from these measurement devices.

2.2. *Nuclear reactor core dynamics*

The dynamical phenomena in a nuclear reactor are occurring within a wide time scale from milliseconds up to months. In the millisecond/second range, the reactor behaviour is strongly influenced by the dynamics of the thermal hydraulics and by the so-called delayed neutrons which are emitted in the order of seconds after the fission takes place. Every reactor is designed to be self-stabilizing with respect to these effects.

On the other end of the time scale we find the slow changes taking place in the reactor, caused by burning up of the uranium fuel as power is being produced. When calculating the effect of burnup, all faster effects are assumed to be in equilibrium.

This paper will be concerned with the medium time scale, and discuss the dynamic phenomena occurring in the hour to day time range. The burn-up effects are then too slow to be of importance, while, on the other hand, equilibrium conditions may be assumed between neutron distribution, thermal hydraulics and delayed neutron concentration. With these limitations, the neutron distribution in the core can be described with sufficient accuracy using the two energy group diffusion equations:

$$\nabla \cdot D_1 \nabla \phi_1 - \Sigma_{a1}\phi_1 - \Sigma_{r1}\phi_1 + \frac{1}{k}(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2) = 0 \tag{1}$$

$$\nabla \cdot D_2 \nabla \phi_2 - \Sigma_{a2}\phi_2 + \Sigma_{r1}\phi_1 = 0 \tag{2}$$

where

ϕ_i is the neutron flux of fast ($i=1$) and slow ($i=2$) neutrons
D_i is the diffusion coefficient in energy group i
Σ_{ai} is the neutron absorption cross section, group i

Σ_{fi} is the fission cross section, group i
Σ_{r1} is the removal cross section from group 1 to group 2
ν is the average number of neutrons emitted per fission

All cross sections are space dependent. The term $\nabla \cdot D_i \nabla \phi_i$ describes the net diffusion of neutrons into the volume element considered, while $\Sigma_{ai}\phi_i$ neutrons are being absorbed in group i per second, and $\Sigma_{r1}\phi_1$ neutrons loose energy and leave energy group 1, entering the low energy group. $\Sigma_{fi}\phi_i$ fissions take place in group i, resulting in the production of $\nu\Sigma_{fi}\phi_i$ neutrons.

These two equations are the conservation laws for neutrons, describing the balance between the terms increasing and the terms reducing the number of neutrons. They are formulated as eigenvalue equations, with an eigenvalue k. When describing a real physical system, the eigenvalue k must be equal to 1. The justification for neglecting the time derivative of the fluxes, and instead introducing the eigenvalue, is that for flux changes in the hour time scale the time derivative terms are many orders of magnitude smaller than the other terms entering the equations.

The cross sections and other parameters change due to the changes in material composition and temperatures in the core caused by thermal-hydraulic changes and control actions changing the control poison distribution (control rod positions and concentration of boric acid in the cooling water).

2.2.1. *Xenon–iodine dynamics*

The dynamic effects in the time scale of hours–days are caused by one of the atoms generated through the decay of the fission products. This atom, xenon, has an extremely large absorption cross section for slow neutrons. In those parts of the core, where the Xe concentration is high, many of the neutrons are absorbed by Xe atoms and the power generation is therefore reduced. The formation of Xe is illustrated in Fig. 1 and the whole xenon–iodine dynamics is governed by the following equations:

$$\frac{dXe}{dt} = -(\lambda_x + \sigma_x \phi_2)Xe + \lambda_I I + \gamma_x S \tag{3}$$

$$\frac{dI}{dt} = -\lambda_I I + \gamma_I S \tag{4}$$

where

Xe, I are the xenon and iodine concentrations
γ_x, γ_I are Xe and I yields from fission
λ_x, λ_I are Xe and I decay constants
$S = \Sigma_{f1}\phi_1 + \Sigma_{f2}\phi_2$ fission rate
$\sigma_x Xe \phi_2$ slow neutron absorption rate in Xe

Xe is mainly formed from the decay of I^{135} with a half-life of 6·7 h. In that time scale I can be considered to come directly from fission. Xe is removed through decay (half life 9·2 h) and absorption. The instantaneous Xe concentration thus depends upon the neutron flux history over the last 30 h or so, and upon the present neutron flux. In a modern light water reactor core the Xe absorption term and decay term are close to each other in magnitude leading to the possibility of Xe induced power oscillations.

If the power for some reason or other would decrease from an equilibrium state, the Xe-concentration would start increasing, because the Xe production through

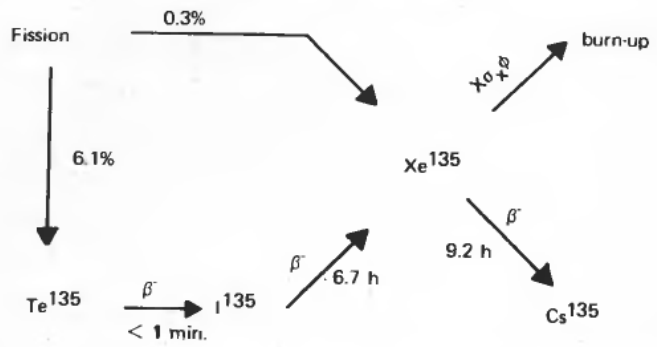

Figure 1. Production of Xe^{135} from fission and by decay of I^{135} and destruction by burn-up and decay.

iodine decay would continue with only a slowly decreasing rate and the removal of xenon through absorption would decrease. The increasing xenon would mean more neutron absorption into xenon instead of fissions and thus a further decrease in the power. Gradually decreasing power would lead to a low iodine concentration and decay and thus to a smaller xenon production. The xenon concentration would then pass a peak value and start decreasing. The decreasing xenon would lead to a diminishing absorption, increasing power and increased iodine production. The power, xenon and iodine would thus start oscillating with typically a 20 to 30 h period. The power and xenon would have about opposite phases, peak power at the time of minimum xenon concentration, while the iodine would follow the power with about 15 degree phase shift.

The effect of xenon in the neutron balance eqns. (1) and (2) occurs in the slow group absorption cross section:

$$\Sigma_{a2}=\Sigma_{a2}{}^{0}+\sigma_{x}Xe \tag{5}$$

where $\Sigma_{a2}{}^{0}$ is the absorption cross section without xenon.

The neutron balance and Xe–I dynamics equations are thus coupled through the fission rate S and the non-linear term $(\sigma_{x}Xe\phi_{2})$.

A further simplification of eqns. (1) and (2) is introduced to reduce the computational effort when calculating the flux distribution in the whole reactor. A reduction from a two to a one energy group problem results if the group 2 neutron diffusion term is neglected. From eqn. (2) the thermal (group 2) flux is expressed:

$$\phi_{2}=\frac{\Sigma_{a2}}{\Sigma_{r1}}\phi_{1} \tag{6}$$

resulting, after some further simplification, in:

$$-M^{2}\nabla^{2}\phi_{1}+\phi_{1}=\frac{k_{\infty}}{k}\phi_{1} \tag{7}$$

where M^2 and k_∞ depend on all the parameters of eqns. (1) and (2). The simplifications were with respect to the spatial dependence operator and require the spatial dependence of the parameters and variables to be sufficiently weak. Physically, the approximations are justified by the fact that the overall flux shape is primarily determined by the fast neutrons, which migrate over much larger distances than the slow neutrons.

2.2.2. *Criticality control*

If the eigenvalue k of eqn. (7) is equal to 1 with the given parameters, the solution represents a real stationary state of the reactor and the reactor is said to be critical. If $k<1$ the real flux would be decreasing with time (the source term was multiplied by $1/k$) and the reactor is said to be subcritical. If $k>1$ the flux time derivatives would be positive and the reactor supercritical.

The criticality control of the reactor means that k (the largest eigenvalue) is controlled to be equal to one. In practice that is done by changing the amount of control absorbers, i.e. control rod banks and boron concentration in the core. Computationally it is simulated by changing the absorption cross sections Σ_{a1} and Σ_{a2} and thereby the parameters M and k of eqn. (7), such that the largest eigenvalue becomes equal to 1.

2.2.3. *Fast responding power feedback effects*

The power density is proportional to the fission rate:

$$P=E_{\text{eff}}(\Sigma_{f1}\phi_1+\Sigma_{f2}\phi_2)$$

$$E_{\text{eff}}\text{: energy release per fission.} \tag{8}$$

Eliminating the thermal flux ϕ_2 from (8) by using eqn. (6), eqn. (7) can be written in terms of the power density. The parameters of eqn. (7) also depend on the power density. Physically one of the feedback effects is due to the fuel temperature and the other due to the coolant temperature. The fuel temperature can be assumed directly proportional to the local power while the coolant temperature also depends on the coolant flow. In order to take into account the coolant temperature effects, the core thermo-hydraulics must be modelled. Both effects have time constants which are much smaller than those of the Xe–I dynamics, and can therefore be treated with their equilibrium values.

The core is designed such that the fuel temperature feedback has a negative feedback coefficient, i.e. tends to diminish the changes in power and thus has a stabilizing effect.

The coolant temperature may have a positive or a negative feedback coefficient changing normally to a more negative value over the core life. The physical mechanism works through the coolant density, which affects the neutron migration and the absorption to soluble boron in the coolant.

2.3. *Spatially dependent core models*

In this section we will go into some more detail of how the spatial dependence of the neutron diffusion equation is treated. The spatial dependence is complex because of the strong heterogeneity of the core.

In order to obtain the parameters (called group constants) of (1) and (2) (or (7)) for the core wide calculations, so-called homogenization calculations must be performed. The core is therefore divided into homogenization regions within which very detailed computations are done. More accurate methods than the diffusion theory are used and many neutron energy groups and a fine spatial mesh are used. The space and speed dependent neutron flux is then used for weighting the space and speed dependent parameters to obtain the group constants. Thereafter the reactor core is assumed regionwise homogeneous, i.e. the parameters constant within the regions.

A straightforward spatial discretization of the resulting three dimensional equations (1) and (2) would, however, require about 10^7 spatial meshpoints. With special methods the number of spatial unknowns can be reduced by several orders of magnitude. The reactor core is divided into subvolumes or nodes, within which the neutron flux is approximated with analytical functions. The average nodal fluxes (or power densities) will then be the unknowns. The number of nodes required is in the order of magnitude of 10^3.

The neutron diffusion equation (7) can be written in the form

$$kP = KLP \tag{9}$$

where P is a vector of the nodal power densities, K is a diagonal matrix with elements $k_{\infty i}$, and L describes the coupling between nodes. For a certain core configuration, nodal division and spatial discretization method, the operators K and L can finally be evaluated as numerical matrices. The power distribution can then be determined using iterative techniques.

Changes in the power distribution are particularly strong when the total power of the core is changed. The increasing share of nuclear plants in an electrical grid will make it necessary for them to follow the daily load variations and perhaps also shorter term variations to stabilize the grid. As the time constants of the Xe–I dynamics are in the same time range, they will play a very important role in the daily operation.

For power level changes both the control rod banks and the boron control are used. When boron is used as controller, the changes in power distribution are smaller than if control rod banks are being used. It is considered advantageous not to perturb the power distribution too much, as large local changes in power may lead to fuel failure. On the other hand, the boron control system has technical limitations on the speed of boron concentration changes, and economical constraints connected to boron dilution. Combined use of the different controllers is thus needed to operate the reactor efficiently.

3. The multistage mathematical programming method

The Multistage Mathematical Programming Method (MMP) has been developed to calculate optimal control for pressurized water reactors. It answers the question, how to move the controllers for the next few hours, when the total reactor power is given and it is desirable to get and keep the power distribution and perhaps some other variables near some given values. Because the controller movements are calculated beforehand, MMP is an open loop method, there is only indirect feedback.

The main control objectives are the desired power distribution and the total power. The control elements are the control rods, which mainly affect the local power density, and the soluble boron concentration in the cooling water. The state variables are the xenon and iodine densities. In this simple model the power density is a dependent variable, that can be calculated, when the xenon density, controller positions, and some other auxiliary variables (as coolant input temperature) are known.

This optimal control problem is described as finding, for some hours ahead in time, the controller movements that minimize a given objective function, while maintaining the total reactor power at desired (variable) level. The objective function describes the deviation of the power distribution, possibly the xenon and iodine distributions, controller positions, etc. from some desired values. Additional requirements are that

some limiting values for power density, controller positions, etc. must not be violated. These requirements are called hard constraints, stressing their strictness.

3.1. *Problem formulation*

The main features of the mathematical formulation of the control problem will be given here. A more detailed description can be found in Karppinen *et al.* (1979).

The optimal control problem in MMP is described as finding the minimum of a given quadratic objective function, subject to linear constraints (quadratic programming formulation). The constrained minimum gives the movements of the controllers over the whole control period. The objective function and constraints are calculated using a linear core model, that is based on a $1\frac{1}{2}$ group formalism reactor simulator code. For this control model the reactor is divided into relatively few (less than a hundred) boxes, called nodes, which may coincide with the nodes of the core simulator, or the control model nodes can be formed by combining several simulator nodes. All variables are averages over these nodes. The state variables are the xenon and iodine densities (e.g. in units atoms/cm^3) and controller positions. The power densities (W/cm^3) are also used, because they are the most important variables to be controlled. The control variables, i.e. the unknowns, are the controller movements. The typically 5 to 10 hour long control period is divided into discrete time steps, typically 0·5 ... 1 hour long. All variables are assumed to change linearly during each time step, and the values for the state variables are taken at the end of each time step. The derivation is made in two parts, first the linear core model is set up, and then the control problem is transformed into quadratic programming form.

3.2. *The linear core model*

To reduce the computational effort in solving the optimization problem, a linearization of the non-linear reactor core model must be introduced. The main non-linearities in the behaviour of a PWR core are the neutron flux (or power)–xenon density product term in the xenon–iodine equation (3), the dependence of the core properties on power through thermal feedback, and the effect of the control rods on the power distribution. The control rod movements change the power densities most near the tips of the rods.

The linear core model consists essentially of two parts. First the dependence of the power density distribution on controller movements, changes in xenon densities etc. is derived, and then the nonlinear xenon–iodine equation is linearized and solved in discrete form, which finally leads to the desired linear model.

3.2.1. *Dependence of power distribution on changes in the core*

Controller movements, changes in xenon density etc. change the power density distribution in the core. The power density is calculated through the infinite multiplication factors (k_∞ in (7)) that in each node depend on the controller movements, xenon density etc. In the control model it is assumed that all changes are slow, the time scale is from tens of minutes to hours. Then the power density distribution can be calculated from the steady state equation in the form of an eigenvalue equation. The equation was given in § 2 (eqn. (9)) and is repeated here:

$$\boldsymbol{kP = KLP} \tag{10}$$

The eqn. (10) is obtained from the diffusion equation (7), by changing it into an integral equation and integrating over the nodes (Locke and Blobel 1974, Silvennoinen 1976).

The components of the eigenvector $\boldsymbol{P}$ are the average power densities in the nodes. K is a diagonal matrix, with elements $k_{\infty i}$ (denoted in the following by k_i), the infinite multiplication factors in the nodes. k_i depends mainly on the neutron absorption properties of the materials in the node. It tells how many neutrons from one fission are available for generating new fissions. k is the eigenvalue, that in steady state is equal to one. The matrix L describes how coupling between nodes affects the power distribution. The coupling comes through neutron currents between nodes. Only coupling to the six nearest neighbours is included in L. The off-diagonal elements of L are:

$$\underset{i\neq j}{L_{ij}}=\frac{M_{ij}}{2h_{ij}}\frac{k_j-k_i}{2k_j}+\frac{{M_{ij}}^2}{{h_{ij}}^2}\frac{k_j+k_i}{2k_j} \tag{11}$$

M_{ij} is the average of the migration lengths M (eqn. (7)) for nodes i and j. k_i and k_j are the infinite multiplication factors for nodes i and j. h_{ij} is the distance between the centres of the nodes i and j. The diagonal element of L are

$$L_{ii}=(1-\sum_j L_{ji}) \tag{12}$$

where the sum goes over the six nearest neighbours of the node i. The reflector nodes which are surrounding the core are not included, but for the diagonal elements (12) the coupling coefficient ${L_{ji}}^r$ for reflector node j may be needed. It is

$${L_{ji}}^r=\frac{M_i}{h_{ij}}\frac{2M_i}{k_i(2\lambda j+h_{ij})} \tag{13}$$

where λ_j is the extrapolation length. It is typically 25 cm.

The dependency of the migration length M on controllers, xenon etc. is negligible, and thus the changes in power density can be calculated through k_i, the infinite multiplication factors in the nodes. To calculate the changes, the eigenvalue equation (10) is differentiated, that is, linearized around some suitable linearization state. The eigenvalue k remains equal to one. This is not a mathematical assumption, but arises from the physical nature of the problem. If some changes are made in the reactor, the power level and power distribution adjust to a level where the eigenvalue k is equal to one. Keeping the eigenvalue k equal to one, when differentiating (10), gives

$$\delta\boldsymbol{P}=\delta(KL)\boldsymbol{P}+KL\delta\boldsymbol{P} \tag{14}$$

The matrix KL depends on changes in power density through thermal feedback, and so the vector $(\delta KL)\boldsymbol{P}$ in (14) depends on $\delta\boldsymbol{P}$. For one component $[(\delta KL)\boldsymbol{P}]_i$ the dependence on small changes in power density $\delta\boldsymbol{P}$ is (index p refers to power feedback)

$$[(\delta KL)\boldsymbol{P}]_{ip}=\sum_l\sum_m\frac{\partial(KL)_{im}}{\partial P_l}P_m\delta P_l=-\sum_l F_{il}\delta P_l \tag{15}$$

when a linear dependence is assumed. F_{il} is the component (il) of the feedback matrix F:

$$F_{il}=-\sum_m\frac{\partial(KL)_{im}}{\partial P_l}P_m \tag{16}$$

The most important feedback arises through the fuel temperature. The moderator temperature effect is much smaller, but through it the power change in one node can affect the properties of other nodes. If this is neglected, assuming that changes in the power density in one node affects only the properties of the node itself, and further the changes in the coupling matrix L are neglected, the feedback matrix F is diagonal. Using (10) its elements are

$$F_{ii} = -\frac{\partial k_i}{\partial P_i}\frac{P_i}{k_i} \tag{17}$$

Also in (14) the changes in the coupling matrix L can be neglected because they are small compared to changes in K. This can be seen from (11). In the control model the nodes are rather large, and the ratio M_{ij}/h_{ij} is small, about 0·1 ... 0·2. With this approximation, using (19) in form $L\boldsymbol{P}=K^{-1}\boldsymbol{P}$ and including the feedback effects, the change in the power density $\boldsymbol{P}$ can be solved from (14):

$$\delta\boldsymbol{P}=(I+F-KL)^{-1}PK^{-1}\delta\boldsymbol{K}_1 \tag{18}$$

where I is the identity matrix. P is now a diagonal matrix, with the same elements as the vector $\boldsymbol{P}$. The elements of the vector $K^{-1}\delta\boldsymbol{K}_1$ are $\delta k_i/k_i$, the relative changes of the infinite multiplication factors in the nodes due to other reasons than thermal feedback. Equation (18) gives thus the changes in power density due to changes in the multiplication factors, which again depend on controller movements, xenon density, etc.

The changes in k_i due to changes in xenon density $\delta\boldsymbol{Xe}$ can be calculated straightforward, $\delta k_i=\partial k_i/\partial Xe_i \cdot \delta Xe_i$, where Xe_i is the xenon density in node i and δXe_i its small change. Thus changes in power density due to xenon are

$$\delta\boldsymbol{P}=(I+F-KL)^{-1}B\delta\boldsymbol{Xe} \tag{19}$$

where B is a diagonal matrix, with elements $B_{ii}=P_i/k_i \cdot \partial k_i/\partial Xe_i$. The power level dependence of the matrix $(I+F-KL)^{-1}B$ through the power densities could be taken approximately into account, but this would complicate the solution of the xenon iodine dynamics equation. As the dependence is rather weak, it is neglected in the model.

3.2.2. *The controllers*

The calculation of the changes in power density due to controller movements is more complicated, because the control rods change the power density most near their tips and the movements of the rods can be large. The main principle is to linearize the effect of the control rod in the node containing the rod tip, and neglect possible changes in other nodes. The change in k_i due to movement u_r (or deviation from the linearization point) of controller r, is

$$\delta k_i=\frac{\partial k_i}{\partial u_r}u_r=\frac{\partial k_i}{\partial c_{ir}}\frac{\partial c_{ir}}{\partial u_r}u_r \tag{20}$$

where the control fraction c_{ir} is introduced (i=node, r=controller), $0\leqslant c_{ir}\leqslant 1$, corresponding rod r out of node i and complete penetrating node i. For several controllers and all nodes the terms $P_i\delta k_i/k_i$ of (18) can be written in matrix form

$$PK^{-1}\delta\boldsymbol{K}_1=[R*G]\boldsymbol{u}=D\boldsymbol{u} \tag{21}$$

where * denotes direct multiplication, i.e. the element (ij) of matrix R is multiplied by the element (ij) of matrix G to get the element (ij) of D. The elements of the matrix R are $R_{ij}=P_i/k_i \cdot \partial k_i/\partial c_{ij}$. G is a so-called geometrical matrix, its elements are $G_{ij}=\partial c_{ij}/\partial u_j$, that are non-zero only in nodes where the rod tip is or where it moves to. In this way the matrix R can be calculated beforehand, and the geometrical matrix G can be adjusted iteratively to correspond to the rod movements.

The matrix product $(I+F-KL)^{-1}D$ (needed in (18)) depends on power, because both F and D contain the power densities. To approximate this dependence, the matrix product can be written

$$(I+F-KL)^{-1}D \approx C_0+C_1P_L \tag{22}$$

where P_L is the average power density in the whole core.

Finally, the changes in the power density due to xenon and controllers can be written, from (18, 19, and 21)

$$\delta \boldsymbol{P}=(I+F-KL)^{-1}B\delta Xe+(C_0+C_1P_L)\boldsymbol{u} \tag{23}$$

Other effects affecting k_i could be included similarly.

3.2.3. *The coupling coefficient matrix L*

MMP is used together with a reactor simulator. If the nodes in the reactor simulator and in the MMP control model coincide, the coupling coefficients matrix L can be obtained directly from the simulator. However, because for accuracy it is desirable to have several small nodes in the simulator, the control model nodes often comprise several simulator nodes. The coupling coefficients must be calculated from the coefficients used in the simulator. The coupling arises from neutron currents between nodes. From (10) and (12) the neutron current J_{ij} between simulator nodes i and j is

$$J_{ij}=V_jL_{ij}P_j-V_iL_{ji}P_i \tag{24}$$

where $V_i=V_j$ are the volumes of the nodes.

The coupling coefficients for the control model are calculated by demanding that neutron current between control model nodes can be expressed similarly as (24) using power densities for the control model nodes, and that the neutron currents are the same in the simulator and in the control model for the linearization state. Another requirement is, that the error in the current in the control model compared to the simulator model is small on average, when the power densities change.

3.3. *Control problem dynamics*

The dynamics of the control problem is governed by the behaviour of the fission products xenon-135 and iodine-135, as described by eqns. (3) and (4). These equations are linearized around the linearization state. In this state the xenon and iodine densities are given by vectors $\boldsymbol{X}_0$ and $\boldsymbol{I}_0$. The state can be a nonequilibrium state, where the time derivatives of the xenon and iodine density vectors $\dot{\boldsymbol{X}}_0$ and $\dot{\boldsymbol{I}}_0$ are non zero. The power density is substituted for the fission rate S and thermal flux Φ_2. There is one to one correspondence between the fission rate and the power density, but for the thermal flux this substitution is only approximate (see eqns. (6) and (8)). The changes

in power density are expressed using eqn. (23). Finally the xenon–iodine equation becomes

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xi} \\ A_{ix} & A_{ii} \end{bmatrix}\begin{bmatrix} x(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} C_{xu}(t) \\ C_{iu}(t) \end{bmatrix} u(t) + \begin{bmatrix} \dot{X}_0 \\ \dot{I}_0 \end{bmatrix} \tag{25}$$

Here $x(t)$ and $i(t)$ are vectors describing the deviations from the linearization state xenon and iodine densities X_0 and I_0. The C_{xu} and C_{iu} matrices are

$$C_{xu} = A_{xu}(C_0 + C_1 P_L(t)), \quad C_{iu} = A_{iu}(C_0 + C_1 P_L(t)) \tag{26}$$

C_1 and C_2 are defined in eqn. (21). All the A-matrices in (25) and (26) can be calculated rather straightforward from eqns. (3, 4, 6, 8, and 23). The components of the control vector $u(t)$ are the controller movements.

The variables giving the power, xenon, and iodine densities are factored into the product of two parts, one giving the average level over the whole core, and the other giving the normalized distributions. For example, $P = P_L p$, where P is the power density vector, P_L the average power density, and p is the normalized power density vector. The xenon and iodine vectors are factored similarly. The calculational state vector is chosen to be of the form (for power here) $P_L \Delta p$, where Δp is the difference between the normalized power distribution as a function of time and the normalized power distribution at the linearization point. The same notation is used for the xenon and iodine vectors. The state vector Z is then, at time step m

$$Z(m) = \begin{bmatrix} P_L \Delta p(m) \\ X_L \Delta x(m) \\ I_L \Delta i(m) \\ u(m) \end{bmatrix} \tag{27}$$

where P_L, X_L and I_L are the power, xenon and iodine levels or average densities, and $u(m)$ gives the controller position deviations from the linearization point. The power level P_L is known, because the total power is assumed to be given. The xenon and iodine levels can then be calculated using a non-linear xenon–iodine equation for the average levels. This equation is obtained by integrating eqns. (3) and (4) over the whole core, assuming that the distributions do not change very much. By this choice of the state variables, the non-linearity of the xenon–iodine dynamics can be taken into account in some way. Also the construction of the linear model becomes slightly simpler when the level factors and distributions are separated.

After substituting the new variables into eqn. (25) it is solved to give the xenon and iodine densities (in the calculational state variables) at time step m, as a function of the xenon and iodine densities at time step $m-1$ and the controller movements during the time step. The solution is only approximate, because a totally accurate solution is too complicated. For details, see (Karppinen *et al.* 1979). Equation (22) is then used

to give the power densities, which completes the derivation of the linear model. The result is:

$$\boldsymbol{Z}(m)=\begin{bmatrix} 0 & G_{px} & G_{pi} & G_{pu} \\ 0 & G_{xx} & G_{xi} & G_{xu} \\ 0 & G_{ix} & G_{ii} & G_{iu} \\ 0 & 0 & 0 & I \end{bmatrix}\boldsymbol{Z}(m-1)+\begin{bmatrix} C_p \\ C_x \\ C_i \\ I \end{bmatrix}\Delta\boldsymbol{u}(m)+\begin{bmatrix} \boldsymbol{g}_p \\ \boldsymbol{g}_x \\ \boldsymbol{g}_i \\ 0 \end{bmatrix} \tag{28}$$

The state vector $\boldsymbol{Z}$ was defined in (26). The components of the vector $\Delta\boldsymbol{u}(m)$ are the movements of the controllers during the time step m. The G matrices and $\boldsymbol{g}$ vectors follow from the derivation outlined above. From (28) it can be seen that the power density is not a real state variable. The linear model can be expressed in the form (28) when the control problem has first been defined, because the G_{pu}, G_{xu} and G_{iu} matrices. C matrices and the $\boldsymbol{g}$ vectors depend on the level factors.

3.4. *Transformation to quadratic programming form*

Using eqn. (28) recursively, the state of the reactor at any time step m can be expressed as a linear function of the initial state and the control movements up to time step m.

The control objective is to get and to keep the reactor state near some desired state at each time step, and to fulfil certain constraints. Both the desired state and the calculated state of the reactor can be expressed using the calculational state variables. The objective function, that measures the distance from the desired states, becomes then

$$J=\sum_{m=1}^{M}(\boldsymbol{Z}(m)-\boldsymbol{Z}_d(m))^T W_m(\boldsymbol{Z}(m)-\boldsymbol{Z}_d(m))+\Delta\boldsymbol{u}_m{}^T R_m\Delta\boldsymbol{u}_m \tag{29}$$

where the sum goes over all time steps from 1 to M. $\boldsymbol{Z}_d(m)$ is the desired state and W_m is a weighting matrix. The latter term in (29) gives also some weight to the controller movements. When the state $\boldsymbol{Z}(m)$ is expressed as a function of only the initial state and the controller movements, the objective function can be written in the standard form

$$J=\boldsymbol{X}^T Q\boldsymbol{X}+C\boldsymbol{X} \tag{30}$$

where Q and C are weighting matrices depending on W_m and R_m in (29). The vector $\boldsymbol{X}$ contains all the controller movements,

$$\boldsymbol{X}^T=(\Delta\boldsymbol{u}^T(1)\ \ldots\ \Delta\boldsymbol{u}^T(M)).$$

Constraints, like limits for the power density, axial imbalance, controller positions, etc. can be directly expressed as linear functions of the controller movements. Constraints for the rate of change of the power density can be approximated by finite differences, and become thus also linear functions of the controller movements. The constraints are then of the form

$$\left.\begin{aligned} A_1\boldsymbol{X}&=\boldsymbol{b}_1 \\ A_2\boldsymbol{X}&\geqslant\boldsymbol{b}_2 \end{aligned}\right\} \tag{31}$$

Thus the control problem has been transformed into that of finding the minimum of a quadratic objective function, eqn. (30), subject to linear constraints, eqn. (31). The

minimum gives the unknown vector X, that contains the controller movements. Standard methods are used in the search for the minimum.

3.5. *The use of MMP*

The use of MMP requires a large amount of computing power in terms of memory and time. If large changes in total power take place during a control period, more than one linear model is needed to ensure high accuracy in the MMP calculation. These models are generated with a reactor simulator using some guessed or previously obtained control sequences. Further, the power demand during the control period must be specified and the initial reactor state known, for example by a state estimator based on measurements. The desired distributions must also be specified. Then the control problem can be formulated as outlined above. The solution of the minimization problem gives the controller movements for the control period. The solution can then be iteratively improved, mainly by better describing the effect of the controllers. This is done by updating the geometrical matrix G in (21). The resulting controller movements are then simulated in a reactor simulator and then either applied or further improved by generating new linear models that correspond better to the expected state of the reactor during the control period.

3.6. *Example*

MMP has not yet been applied to a real reactor. One calculated example is given here. For more examples, see Karppinen and Blomsnes (1976) and Karppinen *et al.* (1979). In this example the reactor is to be controlled over 24 hours with variable power, totally 14 hours at full power, with a 6 hour period at half power, with 2 hour fall and rise times, see Fig. 2. The three-dimensional (1/4 core) reactor model is divided into 36 control nodes, while the non-linear simulator uses 1240 nodes. Two controllers (one control rod bank and boron) are used. Two kinds of control objectives were studied, one to keep the power distribution near the equilibrium distribution ('constant shape') and the other to minimize the use of boron, with some weight also for the power distribution ('minimum boron'). The calculation was made in three control periods of 8 hours, always simulating with the non-linear simulator the control

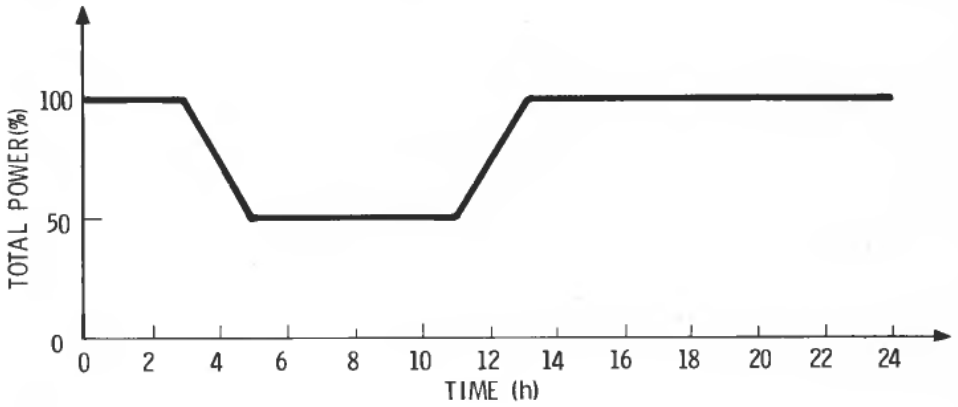

Figure 2. Reactor power as function of time.

movements for the previous control period to get the initial state for the next period. The initial state for the first control period was the equilibrium state.

The results are given in Fig. 3 for the controller movements, and Fig 4 for the axial shape index. The axial shape index is a measure of power distribution (axial shape index = power in lower half of the core minus power in upper half, divided by the total power). In the 'constant shape' control the control rod movements are small, and the power distribution remains rather constant. In the 'minimum boron' control the rods are used extensively and so the power distribution changes much during the control period. As can be seen the control rod movements depend strongly on the relative weights given for the power distribution and the use of boron.

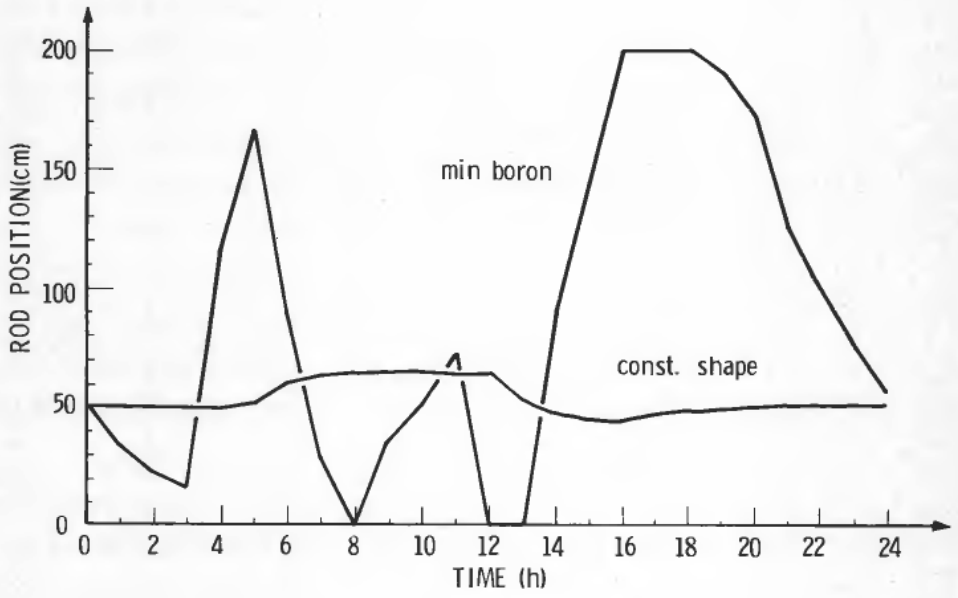

Figure 3. Control bank position as function of time.

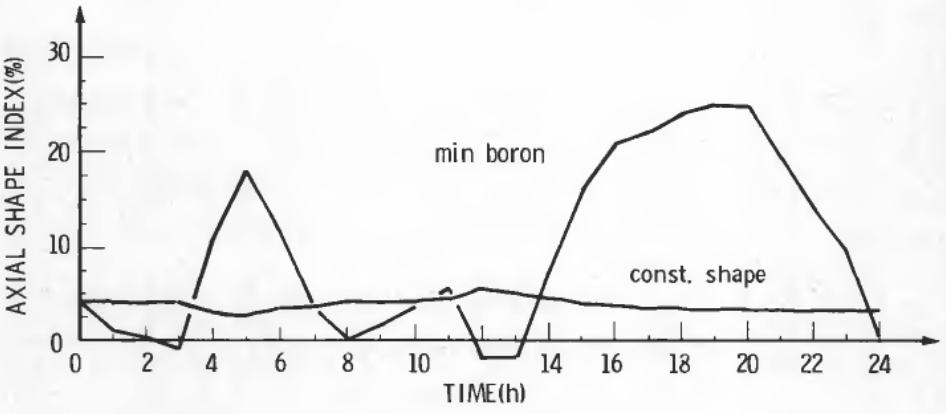

Figure 4. Axial shape index as function of time.

4. Conclusion

The use of Multistage Mathematical Programming for calculation of optimal power distribution control strategies in nuclear reactors has been described. Even if the physical system contains strong non-linearities, the use of more linear models has, through simulation studies, proved to represent the real system well. The present model is believed to be directly applicable for off-line control strategy calculations for pressurized water reactors. Adaptation of the model for on-line application at nuclear plants will be made in the future, if the problem of on-line matrix inversion can be solved for a large number of nodes.

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