Modeling, Identification and Control, Vol. 46, No. 4, 2025, pp. 137–145, ISSN 1890–1328

A Safety Study for Dynamical Systems on Heisenberg Lie Group of dimension 4

S.S. Sütlü [1] B. Akyar [2] A.K. Hansen [3] A. Kıvılcım [4]

[1] *Acibadem University, Faculty of Engineering and Natural Sciences, Department of Computer Engineering, Kayisdagi Street No: 32 Atasehir, Istanbul, Türkiye. E-mail:* sultan.sutlu@acibadem.edu.tr

[2] *IBC International Business College, Dronning Margrethes Vej 6, 6200 Aabenraa, Denmark .E-mail:* bediaakyar@gmail.com; bamo@ibc.dk

[3] *Aarhus University, Department of Mathematics, Ny Munkegade 118, DK-8000, Aarhus C, Denmark. E-mail:* aysekara66@gmail.com; akhansen@au.dk

[4] *Aalborg University, Department of Mathematical Sciences, Thomas Manns Vej 23, 9220 Aalborg Ost, Denmark. E-mail:* akivilcim@gmail.com;ayk@math.aau.dk

Abstract

The safety property of dynamical systems has typically been studied in Euclidean spaces. In this work, we extend the notion of safety to a non-Euclidean geometry. Motivated by the role of time as a fourth dimension in physical models, we construct the 4-dimensional Heisenberg Lie group H_4 and investigate the safety problem of dynamical systems defined on this group. Unlike odd-dimensional Heisenberg Lie groups, which admit a unique structure, even-dimensional cases allow multiple forms; in particular, H_4 possesses four distinct forms. Focusing on one such form, we provide a detailed analysis of dynamical systems on H_4. Moreover, using a diffeomorphism between the (2n+1)-dimensional Heisenberg Lie group and the Euclidean space of the same dimension, we establish their equivalence, and we extend safety result for H_4. Several examples are presented to illustrate the applicability of the theoretical results.

Keywords: Safety, Heisenberg Lie group, dynamical systems

1 Introduction

In this work, we study the safety of dynamical systems on the real Heisenberg Lie group of dimension 4. Safety is a temporal property that has been extensively studied in the context of dynamical systems within Euclidean space. For a given system, safety can be defined as follows: Let χ_0 represent the set of initial states, χ_u the set of unwanted states, and χ the domain. The system is considered safe, if, for all trajectories starting from χ_0, the evolution of the system remains within the domain χ and never enters the set of unwanted states χ_u. Safety has been extensively analyzed in Prajna and Rantzer (2007) for continuous autonomous systems. Moreover, safety has been also investigated for hybrid systems in Prajna and Jadbabaie (2004), for nonlinear switched systems with time dependent switching in Kıvılcım et al. (2019) and for nonlinear switched systems with state dependent switching in Kıvılcım and Wisniewski (2021). However, these aforementioned studies have been conducted in Euclidean spaces. To the best of our knowledge, this work is the first to analyze the safety property in the setting of Heisenberg Lie groups.

Initially, the space in our daily life has been thought to have 3 spatial dimensions, and since Albert Einstein developed his theory of special relativity in Zurich in

doi:10.4173/mic.2025.4.1

1905, the 4th dimension has generally been understood to mean the time. In Lohse et al. (2018) and Zilberberg et al. (2018), the authors have provided the theoretical basis for experiments in which a 4-dimensional physical phenomenon can/could be observed in two dimensions. In addition, the concept of 4 dimensions in astrophysics, three spatial dimensions plus time, is important because it provides a framework for understanding how the universe operates at a fundamental level and the foundation for theories and models that describe the universe's structure, behavior, and evolution, Scano (2024). Motivated by these perspectives, we are inspired to investigate safety properties within the 4-dimensional setting of the Heisenberg Lie group, exploring how dynamical systems behave in this non-Euclidean geometric framework.

This paper consists of six sections. In the second section, we explain in detail the real Heisenberg Lie groups, focusing on the 4th dimension and in the third section, we give the dynamical systems on the real Heisenberg Lie groups of dimension 4, H_4. In the fourth section, we characterize safety of dynamical systems on H_4 and give examples. We present the discussion in the fifth section and lastly the conclusion of the paper.

2 The Heisenberg Lie Group of dimension 4

In this section, we consider Heisenberg Lie groups with real entries. In Székelyhidi (2023) and Colcombet et al. (2019), a generalization of Heisenberg Lie groups is given, where the dimension is $d = 2n+1$ for $n \geq 1$. We explain its particular 4-dimensional case obtained by its succeeding dimension in this section. The Heisenberg Lie group of dimension $2n+1$ has the following form:

$$\mathrm{H}_{2n+1} = \left\{ g = \begin{pmatrix} 1 & \mathbf{a} & \mathrm{c} \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \mid \mathbf{a} \in \mathbb{R}^{1\times n}, \mathbf{b} \in \mathbb{R}^{n\times 1}, \mathrm{c} \in \mathbb{R} \right\},$$

where I_n denotes the $n \times n$ identity matrix.

There is a diffeomorphism $\Psi : \mathrm{H}_{2n+1} \to \mathbb{R}^{2n+1}$ defined by

$$\begin{pmatrix} 1 & \mathbf{a} & \mathrm{c} \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \to (\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_n, \mathrm{b}_1, \mathrm{b}_2, \ldots, \mathrm{b}_n, \mathrm{c}) \in \mathbb{R}^{2n+1}. \quad (1)$$

Thus, it follows that $\mathrm{H}_{2n+1} \cong \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

If we take $n = 2$, then we have the Heisenberg group of dimension 5, denoted by H_5, consisting of 4×4 upper triangular matrices of the form:

$$\mathrm{H}_5 = \left\{ g = \begin{pmatrix} 1 & \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \mathrm{a}_1, \mathrm{a}_2, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c} \in \mathbb{R} \right\}$$

and its Lie algebra has the following form:

$$L(\mathrm{H}_5) = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x_1, x_2, y_1, y_2, z \in \mathbb{R} \right\},$$

where

$$span\left\{ X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

where

$$[X_1, Y_1] = [X_2, Y_2] = Z$$

and the rest of the Lie brackets are null (zero vector).

The Heisenberg group of dimension 4, H_4, is topologically diffeomorphic to $\mathbb{R}^4$ and its structure is slightly different than H_5. It also consists of 4×4 upper triangular matrices of four types of forms obtained from H_5 and we consider the following form:

$$\mathrm{H}_4 = \left\{ g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c} \in \mathbb{R} \right\}$$

and the diffeomorphism $\varphi : \mathrm{H}_4 \to \mathbb{R}^4$ defined by

$$\varphi(\begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}) = (\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}) \in \mathbb{R}^4. \quad (2)$$

H_4 is a simply connected nilpotent Lie group and it has a natural topology induced by its underlying space $\mathbb{R}^4$. Besides, its group structure comes from matrix multiplication and we can write it in terms of $\mathbb{R}^4$ as follows:

$$(a_1, b_1, b_2, c) \star (a_1', b_1', b_2', c') =$$
$$(a_1 + a_1', b_1 + b_1', b_2 + b_2', a_1 b_1' + c + c'), \qquad (3)$$

identity element is $(0,0,0,0)$ and the inverse is

$$(a_1, b_1, b_2, c)^{-1} = (-a_1, -b_1, -b_2, a_1 b_1 - c).$$

Note that $(\mathbb{R}^4, \star)$ is an abstract group and one can see that

$$\forall g_1, g_2 \in H_4, \varphi(g_1 \cdot g_2) = \varphi(g_1) \star \varphi(g_2).$$

Alternatively, H_4 can be described in terms of a Lie algebra structure with the basis elements X_1, Y_1, Y_2, Z, satisfying the nontrivial commutation relations:

$$[X_1, Y_1] = X_1 Y_1 - Y_1 X_1 = Z \quad \textit{and}$$
$$[X_1, Y_2] = [Y_1, Y_2] = [Z, X_1] = [Z, Y_1] = [Z, Y_2] = 0.$$

Then, the Lie algebra has the following form:

$$L(H_4) = \left\{ \begin{pmatrix} 0 & x_1 & 0 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x_1, y_1, y_2, z \in \mathbb{R} \right\},$$

where

$$span\left\{ X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

In fact,

$$[X_1, Y_1] = X_1 Y_1 - Y_1 X_1 =$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = Z$$

and

$$[X_1, Y_2] = X_1 Y_2 - Y_2 X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By similar calculations, $[Y_1, Y_2] = [Z, X_1] = [Z, Y_1] = [Z, Y_2] = 0$. The matrix multiplication of any two vector fields of $L(H_4)$ is

$$\begin{pmatrix} 0 & x_1 & 0 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x_2 & 0 & z \\ 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & y_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} 0 & 0 & 0 & x_1 y_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$\forall X, Y \in L(H_4), d\varphi(X \cdot Y) = d\varphi(g_1) \star' d\varphi(g_2)$$

i.e,

$$\begin{pmatrix} x_1 \\ y_1 \\ y_2 \\ z \end{pmatrix} \star' \begin{pmatrix} x_2 \\ y_3 \\ y_4 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 y_3 \end{pmatrix}, \qquad (4)$$

where $d\varphi$ is the differential of φ at the identity and $\star'$ is the operation between two vector fields of $L(\mathbb{R}^4)$. The derivation algebra $Der(L(H_4))$ of Lie group H_4 is a Lie algebra consisting of endomorphisms D on $L(H_4)$ satisfying

$$\begin{aligned} D[X_1, Y_1] &= [D(X_1), Y_1] + [X_1, D(Y_1)], \\ D[Y_1, Y_2] &= [D(Y_1), Y_2] + [Y_1, D(Y_2)], \\ D[Y_2, Z] &= [D(Y_2), Z] + [Y_2, D(Z)] \quad \text{and} \\ D[Z, X_1] &= [D(Z), X_1] + [Z, D(X_1)]. \end{aligned}$$

In the generalization of Heisenberg Lie groups, dimension is always odd Székelyhidi (2023). On the other hand, if we consider even dimensional cases, then their forms are not unique as we have explained in this section for the dimension 4. In fact, in addition to the form which we use throughout this paper, the other three forms for dimension 4 are

$$\begin{pmatrix} 1 & 0 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $a_1, a_2, b_1, b_2, c \in \mathbb{R}$.

2.1. Corollary:

The $2n$-dimensional Heisenberg Lie group, H_{2n}, has $2n$ different forms for all integers $n \geq 2$.

Proof. Any matrix from H_{2n+1} is of the form

$$\begin{pmatrix} 1 & \mathrm{a}_1 & \cdots & \mathrm{a}_n & \mathrm{c} \\ 0 & 1 & \cdots & 0 & \mathrm{b}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \mathrm{b}_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and H_{2n} can be obtained from H_{2n+1} by assigning zero to just one of $\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_n$ or $\mathrm{b}_1, \mathrm{b}_2, \ldots, \mathrm{b}_n$. It follows that there exist $2n$ different ways to induce H_{2n} from H_{2n+1}. Thus, one can conclude that H_{2n} has $2n$ different forms.

□

Heisenberg Lie groups play an important role in harmonic analysis, quantum mechanics, and several areas of mathematical physics. Our primary intention in this paper is to contribute from theoretical perspective. For this reason, we define dynamical systems on Heisenberg Lie groups.

3 Dynamical Systems on Heisenberg Lie Groups

A continuous-time dynamical system on the state space $\mathbb{R}^n$ is determined by the following equation:

$$\dot{x} = f(x),$$

where $x \in \mathbb{R}^n$ and $f(x)$ is a vector field on $\mathbb{R}^n$. Moreover, for being an abstract group, $\mathbb{R}^n$ is also an Abelian Lie group in which its topology and differentiable manifold structure appear. Thereupon, in this section, we consider dynamical systems on the 4-dimensional Heisenberg group H_4, which is a more general Lie group than $\mathbb{R}^4$.

A dynamical system on a general Heisenberg Lie group H_{2n+1} is determined by the following equation:

$$\dot{g} = X(g), \tag{5}$$

where $g \in \mathrm{H}_{2n+1}$ and X is a smooth vector field on H_{2n+1}. In our work, we consider X as a left-invariant vector field on H_{2n+1}.

3.1. Lemma:

Let $\mathcal{D}_1$ be a dynamical system on $\mathbb{R}^{2n+1}$ defined by $\dot{x} = f(x)$ and let $\mathcal{D}_2$ be a dynamical system on H_{2n+1} defined by $\dot{g} = X(g)$, where f is a left-invariant vector field on $\mathbb{R}^{2n+1}$ and X is an element from the Lie algebra of H_{2n+1}. Then, the dynamical systems $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent which is denoted by $\simeq$.

Proof. Assume that the dynamical systems are not equivalent, which means that there is no diffeomorphism between $\mathcal{D}_1$ and $\mathcal{D}_2$, Iliashenko and Li (1999). It follows that there is no diffeomorphism between f and X. This implies that there is no diffeomorphism between $\mathbb{R}^{2n+1}$ and H_{2n+1}, which contradicts with (1). Thus,

$$\dot{x} = f(x) \quad \simeq \quad \dot{g} = X(g).$$

□

3.2. Corollary:

Let $\dot{x} = f(x)$ define a dynamical system on $\mathbb{R}^4$ and $\dot{g} = X(g)$ define a dynamical system on H_4. Then, these two dynamical systems are equivalent.

Proof. Let $\pi_{1,3}$ be the projection of H_5 to H_4 such that for every $g \in \mathrm{H}_5$, $\pi_{1,3}(g) = g' \in \mathrm{H}_4$ assigning 0 to the entry $\mathrm{a}_2 \in g$, and let π_2 be the projection of $\mathbb{R}^5$ to $\mathbb{R}^4$ such that for every $(\mathrm{a}_1, \mathrm{a}_2, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}) \in \mathbb{R}^5$,

$$\pi_2(\mathrm{a}_1, \mathrm{a}_2, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}) = (\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}) \in \mathbb{R}^4$$

Then, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_5 & \xrightarrow{\Psi_5} & \mathbb{R}^5 \\ \downarrow{\scriptstyle \pi_{1,3}} & & \downarrow{\scriptstyle \pi_2} \\ H_4 & \xrightarrow{\varphi} & \mathbb{R}^4 \end{array}$$

where $\varphi : H_4 \to \mathbb{R}^4$ is the diffeomorphism given in the previuos section and $\Psi_5 : \mathrm{H}_5 \to \mathbb{R}^5$ is the diffeomorphism defined by

$$\Psi_5(\begin{pmatrix} 1 & \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}) = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}) \in \mathbb{R}^5.$$

Hence, by using Lemma 3.1 for $n = 2$, the proof is complete. □

3.3. Lemma:

Let us consider Lie groups $\mathbb{R}^4$ and H_4 with their Lie algebras $L(\mathbb{R}^4)$ and $L(\mathrm{H}_4)$, respectively. Denote by $\exp_{\mathbb{R}^4}$ the exponential map from $L(\mathbb{R}^4)$ to $\mathbb{R}^4$ and by $\exp_{\mathrm{H}_4}$ the exponential map from $L(\mathrm{H}_4)$ to H_4. Then, the following diagram is commutative

$$\begin{array}{ccc} L(\mathbb{R}^4) & \xrightarrow{\exp_{\mathbb{R}^4}} & \mathbb{R}^4 \\ \uparrow d\varphi & & \uparrow \varphi \\ L(\mathrm{H}_4) & \xrightarrow{\exp_{\mathrm{H}_4}} & \mathrm{H}_4 \end{array}$$

where $\varphi : \mathrm{H}_4 \to \mathbb{R}^4$ is a Lie group homomorphism and $d\varphi$ is its differential at the identity.

Proof. We want to prove that $\exp_{\mathbb{R}^4} \circ d\varphi = \varphi \circ \exp_{\mathrm{H}_4}$, where $\circ$ denotes the standard composition of two functions. Note that $\mathbb{R}^4$ is an abelian Lie group, H_4 is a nilpotent Lie group and both groups are simply connected. Then, the exponential maps are global diffeomorphisms.

$$\forall X = \begin{pmatrix} 0 & x_1 & 0 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L(\mathrm{H}_4),$$

$$\exp_{\mathrm{H}_4}(tX) = \begin{pmatrix} 1 & tx_1 & 0 & tz + \frac{t^2 x_1 y_1}{2!} \\ 0 & 1 & 0 & ty_1 \\ 0 & 0 & 1 & ty_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4,$$

where $t \in \mathbb{R}$ and

$$\varphi(\begin{pmatrix} 1 & tx_1 & 0 & tz + \frac{t^2 x_1 y_1}{2!} \\ 0 & 1 & 0 & ty_1 \\ 0 & 0 & 1 & ty_2 \\ 0 & 0 & 0 & 1 \end{pmatrix})$$

$$= (tx_1, ty_1, ty_2, tz + \frac{t^2 x_1 y_1}{2}) \in \mathbb{R}^4.$$

On the other hand,

$$d\varphi(X) = d\varphi(\begin{pmatrix} 0 & x_1 & 0 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} x_1 \\ y_1 \\ y_2 \\ z \end{pmatrix} \in L(\mathbb{R}^4),$$

$$\begin{pmatrix} 0 & x_1 & 0 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x_1 & 0 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & x_1 y_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and X^3 is 4×4 zero matrix.

Any $X = \begin{pmatrix} x_1 \\ y_1 \\ y_2 \\ z \end{pmatrix} \in L(\mathbb{R}^4)$ defines a straight line, then by using (4) for Lie algebra elements and writing outputs of exponential map as elements of Lie group, we have

$$\exp_{\mathbb{R}^4}(t \begin{pmatrix} x_1 \\ y_1 \\ y_2 \\ z \end{pmatrix}) = \mathrm{e}^{\begin{pmatrix} tx_1 \\ ty_1 \\ ty_2 \\ tz \end{pmatrix}} = \sum_{n=0}^{\infty} \frac{\begin{pmatrix} tx_1 \\ ty_1 \\ ty_2 \\ tz \end{pmatrix}^n}{n!}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T + \begin{pmatrix} tx_1 \\ ty_1 \\ ty_2 \\ tz \end{pmatrix}^T + \frac{t^2}{2!}(\begin{pmatrix} x_1 \\ y_1 \\ y_2 \\ z \end{pmatrix}^2)^T + \mathbf{0} + \dots$$

$$= (0,0,0,0) + (tx_1, ty_1, ty_2, tz) + \frac{(0,0,0,t^2 x_1 y_1)}{2!}$$

$$= (tx_1, ty_1, ty_2, tz + \frac{t^2 x_1 y_1}{2}) \in \mathbb{R}^4,$$

where, $\begin{pmatrix} tx_1 \\ ty_1 \\ ty_2 \\ tz \end{pmatrix} \star' \begin{pmatrix} tx_1 \\ ty_1 \\ ty_2 \\ tz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 x_1 y_1 \end{pmatrix}$. □

3.4. Example:

Let us consider the dynamical system on H_4 governed by the following differential equation:

$$\dot{g} = \begin{pmatrix} 0 & 5 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We know that $\mathrm{H}_4 \cong \mathbb{R}^4$ and $L(\mathrm{H}_4) \cong L(\mathbb{R}^4)$. Then,

$$\dot{g} = \begin{pmatrix} 0 & 5 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \simeq \dot{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 3 \end{pmatrix},$$

where $g \in \mathrm{H}_4$ and $x \in \mathbb{R}^4$.

4 Safety for Dynamical Systems on Heisenberg Lie Groups

In this section, we study the safety of dynamical systems in the state space H_4, where the vector fields X are from its Lie algebra. In order to do that, we will provide a definition of safety for Heisenberg group of dimension 4.

Safety is temporal property widely investigated for dynamical systems in Euclidean space Prajna and Rantzer (2007). Safety can be defined for a given system as follows: For a given set of initial states, χ_0, unwanted states, χ_u and domain, χ the system is called safe if the solutions starting from the set of inital states do not enter the unwanted states as long as they stay in the domain. Based on our current understanding, this may be the first time in literature that the safety property has been analyzed for Heisenberg Lie groups. The following definition is adapted from Prajna and Rantzer (2007) and generalized to dynamical systems on Heisenberg Lie groups.

4.1. Definition:

For a given domain $\chi \subset H_4$, a set of initial states $\chi_0 \subset H_4$ and a set of unsafe states $\chi_u \subset H_4$, it can be said that the system (5) is safe if there is no solution $g(t)$ of system (5) such that $g(0) \in \chi_0$, $g(T) \in \chi_u$ and $g(t) \in \chi$, for some $T > 0$ and for all $t \geq 0$.

Next, a result will be provided to verify the safety of the Heisenberg Lie group of dimension 4, H_4 by being inspired from the technique given in Prajna and Rantzer (2007).

4.2. Theorem:

Let us consider the following dynamical system on the Heisenberg Lie group of dimension 4, H_4:

$$\dot{g(t)} = (Y_i)(g(t)), \tag{6}$$

where $Y_i \in L(H_4)$, and let the sets $\chi \subseteq H_4$, $\chi_0 \subseteq H_4$ and $\chi_u \subseteq H_4$ be given. If there exists a function $B \in C^1(\mathbb{R}^4)$ satisfying

$$B(\varphi(g)) \leq 0, \ \forall g \in \chi_0, \tag{7}$$

$$B(\varphi(g)) > 0, \ \forall g \in \chi_u \tag{8}$$

$$d(B \circ \varphi)(g)d\varphi(Y_i) = \nabla(B(\varphi(g)))d\varphi(Y_i) \leq 0, \quad \forall g \in \chi \subseteq H_4 \tag{9}$$

then the system (6) is safe with the given domain, the set of initial states and the set of unwanted states.

Proof. Let us consider that there exists a function $B \in C^1(\mathbb{R}^4)$ satisfying (7), (8) and (9) and that the dynamical system (6) is not safe. This implies that there exists a time $T \geq 0$ and an initial state $g_0 \in \chi_0$ such that a trajectory $g(t)$ of the system starting at $g(0) = g_0$ satisfies $g(t) \in \chi \subseteq H_4$ for all $t \in [0, T]$ and $g(T) \in \chi_u$ for $T \geq 0$. Note that for all initial states $g_0 \in \chi_0$, we have $B(\varphi(g)) \leq 0$ from (7). Recall that $(B \circ \varphi)(g(t)) = B(x(t))$ and taking the time derivative of both sides, we get

$$\begin{aligned} \frac{d(B\circ\varphi)(g(t))}{dt} &= \frac{dB(x(t))}{dt} = \nabla B(x(t)) \cdot \dot{x}(t) \\ &= \nabla B(\varphi(g(t))) \cdot d\varphi(\dot{g}(t)) \\ &= \nabla B(\varphi(g(t))) \cdot d\varphi(Y_i). \end{aligned}$$

Utilizing the mean value theorem, we have

$$\frac{B(\varphi(g(T))) - B(\varphi(g(0)))}{T} = \frac{d(B(\varphi(g(t)))}{dt}\,|_{t=T'},$$

where $T' \in (0, T)$. Using (9) together with the above equality, we get

$$\frac{B(\varphi(g(T))) - B(\varphi(g(0)))}{T} = \frac{d(B\circ\varphi)(g(t))}{dt}\,|_{t=T'}$$

$$= \nabla B(\varphi(g(T')))d\varphi(Y_i) \leq 0.$$

Using (7), the previous inequality follows that $B(\varphi(g(T))) \leq B(\varphi(g(0))) \leq 0$ and this contradicts (8). Therefore, there exists no solution starting from χ_0 that reaches χ_u as long as it stays in χ. $\square$

4.3. Example:

Let us consider the dynamical system on H_4 governed by the following differential equation:

$$\dot{g} = \begin{pmatrix} 1 & a_1 & 0 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & \cos(k) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

where k is a constant. Choose a function $B \in C^1(\mathbb{R}^4)$ defined by

$$B(a_1, b_1, b_2, c) = -\sin(b_1) - \sin(b_2) - \sin(c)$$

and define

$$\chi_0 = \left\{ g = \begin{pmatrix} 1 & a_1 & 0 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_4 \mid a_1, b_1, b_2, c \in [0, \pi] \right\}$$

and

$$\chi_u = \left\{ g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4 \mid \right.$$

$$\left. \mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c} \in [\frac{7\pi}{6}, \frac{11\pi}{6}] \right\}.$$

Then, $\forall g \in \chi_0$,

$$B(\varphi(g)) \leq 0$$

which satisfies (7). Besides, $\forall g \in \chi_u$,

$$B(\varphi(g)) > 0$$

which satisfies (8). Finally, $\forall g \in \chi \subseteq \mathrm{H}_4$,

$$d(B \circ \varphi)(g)) = \nabla(B(\varphi(g))) = \begin{pmatrix} 0 \\ -\cos(\mathrm{b}_1) \\ -\cos(\mathrm{b}_2) \\ -\cos(\mathrm{c}) \end{pmatrix}$$

and

$$d\varphi(Y_i) = d\varphi = \begin{pmatrix} 0 & \cos(\mathrm{k}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\mathrm{k}) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The dot product of these two vectors is

$$\nabla(B(\varphi(g))) \cdot d\varphi(Y_i) = \begin{pmatrix} 0 \\ -\cos(\mathrm{b}_1) \\ -\cos(\mathrm{b}_2) \\ -\cos(\mathrm{c}) \end{pmatrix} \cdot \begin{pmatrix} \cos(\mathrm{k}) \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

which verifies condition(9). Since the conditions (7)-(9) are satisfied for the chosen $B(\varphi(g))$ and the sets χ_0, χ_u and χ, then we can conclude that the system is safe.

4.4. Example:

Let us consider the dynamical system on H_4 governed by the following differential equation:

$$\dot{g} = \dot{\begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\kappa} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

where κ is a constant. Choose a function $B \in C^1(\mathbb{R}^4)$ defined by

$$B(\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}) = \mathrm{a}_1^2 + \mathrm{b}_1^2 + \mathrm{b}_2^2 + \mathrm{c}^2 - \mathrm{R}^2$$

for $R > 0$ is any constant and define

$$\chi_0 = \left\{ g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4 \mid \right.$$

$$\left. \mathrm{a}_1^2 + \mathrm{b}_1^2 + \mathrm{b}_2^2 + \mathrm{c}^2 \leq r_0^2 \text{ for } 0 < r_0 \leq R \right\}$$

and

$$\chi_u = \left\{ g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4 \mid \right.$$

$$\left. \mathrm{a}_1^2 + \mathrm{b}_1^2 + \mathrm{b}_2^2 + \mathrm{c}^2 > R^2 \right\}.$$

Then, $\forall g \in \chi_0$,

$$B(\varphi(g)) = \mathrm{a}_1^2 + \mathrm{b}_1^2 + \mathrm{b}_2^2 + \mathrm{c}^2 - R^2 \leq r_0^2 - R^2 \leq 0$$

which satisfies (7). Besides, $\forall g \in \chi_u$,

$$B(\varphi(g)) = \mathrm{a}_1^2 + \mathrm{b}_1^2 + \mathrm{b}_2^2 + \mathrm{c}^2 - R^2 > R^2 - R^2 = 0$$

which satisfies (8). Finally, $\forall g \in \chi \subseteq \mathrm{H}_4$,

$$d(B \circ \varphi)(g)) = \nabla(B(\varphi(g))) = \begin{pmatrix} 2\mathrm{a}_1 \\ 2\mathrm{b}_1 \\ 2\mathrm{b}_2 \\ 2\mathrm{c} \end{pmatrix}$$

and

$$d\varphi(Y_i) = d\varphi \begin{pmatrix} 0 & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\kappa} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\kappa \\ 0 \\ \mathrm{e}^{\kappa} \\ 0 \end{pmatrix}.$$

The dot product of these two vectors is

$$\nabla(B(\varphi(g))) \cdot d\varphi(Y_i) = \begin{pmatrix} 2\mathrm{a}_1 \\ 2\mathrm{b}_1 \\ 2\mathrm{b}_2 \\ 2\mathrm{c} \end{pmatrix} \cdot \begin{pmatrix} -\kappa \\ 0 \\ \mathrm{e}^{\kappa} \\ 0 \end{pmatrix} = -2\mathrm{a}_1\kappa + 2\mathrm{b}_2\mathrm{e}^{\kappa}.$$

If $\mathrm{a}_1\kappa - \mathrm{b}_2\mathrm{e}^{\kappa} \geq 0$, then the system is safe.

4.5. Example:

Let us consider the dynamical system on H_4 governed by the following differential equation:

$$\dot{g} = \dot{\begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

with $\chi = \mathrm{H}_4$, $\chi_0 \subset \mathrm{H}_4$ and $\chi_u \subset \mathrm{H}_4$.
By the diffeomorphism φ between H_4 and $\mathbb{R}^4$, we can consider the same dynamic as

$$\dot{x} = (1, 1, 1, 1)^T,$$

with $\chi_{\mathbb{R}^4} = \mathbb{R}^4$, $\chi_{0,\mathbb{R}^4} = [1, 2] \times [1, 2] \times [1, 2] \times [1, 2]$ and $\chi_{u,\mathbb{R}^4} = [-2, -1] \times [-2, -1] \times [-2, -1] \times [-2, -1]$. Then,

$$\chi = \varphi^{-1}(\mathbb{R}^4) = \{g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4 \mid$$

$$\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c} \in \mathbb{R}\},$$

$$\chi_0 = \varphi^{-1}(\chi_{0,\mathbb{R}^4}) = \{g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4 \mid$$

$$\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c} \in [1, 2]\} \subset \mathrm{H}_4$$

and

$$\chi_u = \varphi^{-1}(\chi_{u,\mathbb{R}^4}) = \{g = \begin{pmatrix} 1 & \mathrm{a}_1 & 0 & \mathrm{c} \\ 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 0 & 1 & \mathrm{b}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{H}_4 \mid$$

$$\mathrm{a}_1, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c} \in [-2, -1]\} \subset \mathrm{H}_4.$$

For $B(\varphi(g)) = -\mathrm{a}_1 - \mathrm{b}_1 - \mathrm{b}_2 - \mathrm{c}$, we can see that $B(\varphi(g)) < 0$, for all $g \in \chi_0$, and $B(\varphi(g)) > 0$, for all $g \in \chi_u$, which verifies the conditions (7) and (8), respectively. Condition (9) can be verified by observing

$$d(B \circ \varphi)(g) d\varphi(Y_i) = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = -4 < 0.$$

Thus, the conditions of Theorem 4.2 are satisfied, it can be concluded that the system (12) is safe with the given sets.

4.6. Remark:

Some generalization of the safety verification technique given in Theorem 4.2 for H_4, can also be provided for Heisenberg groups of dimensions $2n$ and $2n+1$, $n \geq 1$. To provide a generalization for H_{2n+1}, the standard diffeomorphism Ψ, given as (1), and Lemma 3.1 can be used to show the equivalence between dynamical system on H_{2n+1} and $\mathbb{R}^{2n+1}$. More precisely, in Theorem 4.2 replacing φ and H_4 with Ψ and H_{2n+1}, respectively, a generalization of Theorem 4.2 to H_{2n+1} can be obtained. Similarly, the result of Theorem 4.2 can be generalized to H_{2n}, by obtaining H_{2n} from H_{2n+1} as mentioned in Corollary 2.1 and using a suitable diffeomorphism.

5 Discussion

Safety is a temporal property that has been studied extensively in the context of dynamical systems in Euclidean spaces. Traditionally, the space we experience in everyday life is considered to have three spatial dimensions. The concept of 4 dimensions in astrophysics, three spatial dimensions plus time, is important because it provides a frame-work for understanding how the universe operates at a fundamental level and the foundation for theories and models that describe the universe's structure, behavior, and evolution, Scano (2024). This leads us to the importance of 4-dimensional dynamical system models and looking for new models. Heisenberg Lie group is topologically diffeomorphic to the Euclidean space and therefore it is interesting to study safety problem of the dynamical systems on Heisenberg Lie groups of dimension 4, H_4, exploring how dynamical systems behave in this non-Euclidean geometric framework. In the literature of Lie groups, generalization of Heisenberg Lie groups is given for odd dimensions. When the dimensions are even, we point out that they do not have a unique form as they do in odd dimension. In fact, H_4 has four different forms. Therefore, we study in detail for dimension 4 by considering one of its four forms. We construct dynamical systems on H_4 and characterize the safety of them. We focus on dimension 4 from its possible applications point of view.

6 Conclusion

In this work, we first provided some properties of the Heisenberg Lie group of dimension 4 which was obtained from the Heisenberg Lie group of dimension 5 by using proper projections. Moreover, using a diffeomorphism between the Euclidean space $\mathbb{R}^{2n+1}$ and the Heisenberg Lie group H_{2n+1}, we proved that they were equivalent. We also showed that the Euclidean space $\mathbb{R}^4$ and the Heisenberg Lie group H_4 are equivalent similarly. Furthermore, we mentioned the safety property for Euclidean spaces and we have generalized it to Heisenberg Lie groups with dimensions 4 and $2n+1$, respectively. In addition, we provided some examples to show the applicability of the theoretical results.

References

Colcombet, T., Ouaknine, J., Semukhin, P., and Worrell, J. On Reachability Problems for Low-Dimensional Matrix Semigroups. 2019. pages 44:1–44:15. doi:10.4230/LIPIcs.ICALP.2019.44.

Iliashenko, I. U. S. and Li, W. Nonlocal bifurcations. 1999. doi:10.1090/surv/066.

Kıvılcım, A., Karabacak, , and Wisniewski, R. Safety verification of nonlinear switched systems via barrier functions and barrier densities. In *2019 18th European Control Conference (ECC)*. pages 776–780, 2019. doi:10.23919/ECC.2019.8796081.

Kıvılcım, A. and Wisniewski, R. Weak safe reachability for nonlinear systems with state-dependent switching. In *2021 60th IEEE Conference on Decision and Control (CDC)*. pages 5350–5355, 2021. doi:10.1109/CDC45484.2021.9683662.

Lohse, M., Schweizer, C., Price, H. M., Zilberberg, O., and Bloch, I. Exploring 4d quantum hall physics with a 2d topological charge pump. *Nature*, 2018. 553(7686):55–58. doi:10.1038/nature25000.

Prajna, S. and Jadbabaie, A. Safety verification of hybrid systems using barrier certificates. In R. Alur and G. J. Pappas, editors, *Hybrid Systems: Computation and Control*. Springer Berlin Heidelberg, Berlin, Heidelberg, pages 477–492, 2004. doi:10.1007/978-3-540-24743-2$_3$2.

Prajna, S. and Rantzer, A. Convex programs for temporal verification of nonlinear dynamical systems. *SIAM Journal on Control and Optimization*, 2007. 46(3):999–1021. doi:10.1137/050645178.

Scano, P. S. Space and time, do we describe them correctly? on the 4-dimensional structure of the universe. *Nature*, 2024. doi:10.33774/coe-2024-203js.

Székelyhidi, L. A generalization of the heisenberg group. *Journal of the Iranian Mathematical Society*, 2023. 4(2):105–119. doi:10.30504/jims.2023.396137.1113.

Zilberberg, O., Huang, S., Guglielmon, J., Wang, M., Chen, K. P., Kraus, Y. E., and Rechtsman, M. C. Photonic topological boundary pumping as a probe of 4d quantum hall physics. *Nature*, 2018. 553(7686):59–62. doi:10.1038/nature25011.