Modeling, Identification and Control, Vol. 40, No. 1, 2019, pp. 41–50, ISSN 1890–1328

Master-slave $\mathcal{H}_\infty$ robust controller design for synchronization of chaotic systems

Vojtech Veselý [1] Adrian Ilka [2] Ladislav Körösi [1] Martin Ernek [1]

[1] *Institute of Robotics and Cybernetics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology. E-mail:* {vojtech.vesely, ladislav.korosi, martin.ernek}@stuba.sk

[2] *Department of Electrical Engineering, Chalmers University of Technology, Hörsalsvägen 9-11, SE-412 96, Gothenburg, Sweden. E-mail:* adrian.ilka@chalmers.se

Abstract

This paper is devoted to robust master-slave controller design for generalized chaotic systems synchronization. The closed-loop system is asymptotically stable when the robust stability conditions hold and while the $\mathcal{H}_\infty$ norm of the closed-loop transfer function with respect to defined output and input is strictly less than $\gamma > 0$. In this paper a modified $\mathcal{L}_2$ gain approach is used and an original design procedure is proposed to decrease the conservativeness of the former method. The effectiveness of the proposed method is shown in numerical examples.

Keywords: Chaotic systems, master-slave systems, $\mathcal{H}_\infty$ robust controller, Synchronization of chaotic systems, $\mathcal{L}_2$ gain performance.

1 Introduction

The chaotic synchronization gets much more attention due to the powerful applications in biological systems, chemical reactions, information processing, power converters, secure communications and others. The mentioned topic has attracted many researchers since the original paper by Pecora and Carroll (1990) dedicated to chaotic synchronization and it still belongs to the interesting problems in nonlinear oscillator control. The main idea of synchronization is to design the controller for a slave chaotic system, that the slave output can follow the master system output while the defined error is asymptotically stable, (Wang et al., 2012). Chaos synchronization opens up huge perspective to optimize nonlinear oscillated systems. Electrical power system belongs to this type of nonlinear dynamic systems. Study of the chaos and its control (Shahverdiev et al., 2008) could avoid undesirable power system dynamic and behaviour, which should lead to the power system blackout. One of the essences of the proposed method mentioned also in the paper Song and Yu (2003) is that the tracking error of master-slave system is pushed and forced to the pre-selected invariant manifold. The necessary chaos synchronization condition is that the conditional Lyapunov exponents are negative. Robust $\mathcal{H}_\infty$ controller design for synchronization of master-slave dynamic chaotic systems using LMI and feedback linearization has been previously made in Wang and Balakrishnan (2002) and further results to synchronization of Lure systems with time delay in Zeng et al. (2015). The slave control algorithm has been given for the case of fourth order chaotic system with rather complicated control algorithm. Chaos synchronization of Rossler systems, that has origins in chemical kinetics is described in Farghaly (2013). The controller used to synchronize two identical Rossler dynamic systems has been determined by the Lyapunov function. The sufficient global stability criterion for synchronizing Liu chaotic system has been obtained based on a linear feedback control method and Lyapunov function in Chen (2009). Disturbance observer based controller

doi:10.4173/mic.2019.1.4

could be found in Mobayen and Javadi (2015). Chaotic system control with Lipschitz nonlinearities and state feedback is described in Mobayen and Tchier (2017). New chaotic regimes in the Lorenz and Chen systems are described in the paper Sprott (2015). The paper describes new regimes and show that Lorenz and Chen chaotic systems admits chaotic solution. The review of chaotic systems and synchronization control should also be consulted in Boccaletti et al. (2002) and excellent book of L Fradkov (2007). On the basis of the above small observation the following problem is studied in this paper: Develop a new design procedure for the given dynamics of chaotic system (8) to design the robust $\mathcal{H}_\infty$ controller based on the LMI, feedback linearization and bounded real lemma with less conservativeness than known methods from references using bilinear matrix inequalities (BMIs).

The contribution of this paper is to provide robust performance conditions by minimization of $\mathcal{H}_\infty$ gain with respect to the calculated output and disturbance input for the feedback interconnection of a chaos slave system and designed controller which ensures the synchronization in master-slave systems.

The remainder of the paper is organized as follows. In Section 2 we present preliminaries and recall design procedure for $\mathcal{H}_\infty$ gain. In Section 3 we address the state error tracking feedback to obtain the new less conservative robust $\mathcal{H}_\infty$ controller design procedure. Finally, in Section 4, the proposed design procedure is verified by the examples.

Our notation used in the paper is standard, $P \in R^{m\times n}$ denotes the set of real $m \times n$ matrices, $P \succ 0$ $(P \succeq 0) \in R^{n\times n}$ is a real symmetric, positive definite (semidefinite) matrix. Furthermore, "*" in matrices denotes the respective transposed (conjugate) term to make matrix symmetric. Finally, I_m is an $m \times m$ identity matrix, and 0_m is an $m \times m$ zero matrix.

2 Preliminaries and problem formulation

Consider the following chaotic system (Wang et al., 2012) in the form

$$\begin{aligned} \dot{q}_i &= q_{i+1}, \quad i = 1, 2, \ldots, n-1 \\ \dot{q}_n &= f(q, w, v), \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^n$ is the state vector, $v \in \mathbb{R}^m$ is the control input vector, and $w \in R^k \in L_2[t_0, \infty)$ is an exogenous input vector. The system (1) one can split to slave system

$$\begin{aligned} \dot{q}_{s_i} &= q_{s_{i+1}}, \quad i = 1, 2, \ldots, n-1 \\ \dot{q}_{s_n} &= f_1(q_s) + w + u, \end{aligned} \tag{2}$$

and to master closed-loop system

$$\begin{aligned} \dot{q}_{m_i} &= q_{m_{i+1}}, \quad i = 1, 2, \ldots, n-1 \\ \dot{q}_{m_n} &= f_2(q_m), \end{aligned} \tag{3}$$

where $f_i(.)$, $i = 1, 2$ are nonlinear function variables, wherein $q_s \in \mathbb{R}^n$ and $q_m \in \mathbb{R}^n$ are the state vectors of slave and master chaotic systems, and $u \in \mathbb{R}^m$ is the control input vector of the slave system. The slave (2) and master (3) systems can be transformed to matrix form as

$$\dot{q}_s = A_s q_s + Bu + B_1 w + A_{sn}(q_s), \tag{4}$$

$$\dot{q}_m = A_m q_m + A_{mn}(q_m), \tag{5}$$

where

$$A_s = A_m = \begin{bmatrix} 0, & 1, & 0 & \cdots & 0 \\ 0, & 0, & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0 & \cdots & 0 \end{bmatrix}, \; B = B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

$$A_{sn} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f_1(q_s) \end{bmatrix}, \; A_{mn} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f_2(q_m) \end{bmatrix}.$$

Such models can represent many types of chaotic systems. The above mentioned idea of two chaotic systems synchronization is to design the controller for slave system such that the defined output (states) of the slave system can follow the corresponding output (states) of master systems. That implies that the tracking error for both systems will be asymptotically stable

$$\lim_{t\to\infty} ||q_s - q_m|| \to 0. \tag{6}$$

The tracking error is defined as

$$e = q_s - q_m. \tag{7}$$

Subtracting equations (4) and (5) one obtains

$$\begin{aligned} \dot{e} &= Ae + B_1 w + Bu + BF(q_s, q_m), \\ z &= C_1 e + D_{11} w + D_{12} u + D_{12} F(q_s, q_m), \\ y &= Ce + D_{21} w, \end{aligned} \tag{8}$$

where $z \in \mathbb{R}^{l_z}$ is the performance output vector, $y \in \mathbb{R}^l$ is the measurable output vector, $A = A_s$, and $F(q_s, q_m) = f_1(q_s) - f_2(q_m)$. The matrices $C \in \mathbb{R}^{l\times n}$, $C_1 \in \mathbb{R}^{l_z\times n}$, $D_{11} \in \mathbb{R}^{l_z\times k}$, $D_{12} \in \mathbb{R}^{l_z\times m}$, and $D_{21} \in \mathbb{R}^{l\times k}$ are known constant matrices.

Let us recall some standard terminology.

Definition 2.1. **($\mathcal{L}_2$ norm, Boyd et al. (1994))** The $\mathcal{L}_2$ norm of $h \in \mathbb{R}^n$ is defined as:

$$\|h\|_2^2 = \int_0^\infty h^T h \, dt. \tag{9}$$

Definition 2.2. **($\mathcal{L}_2$ gain, Boyd et al. (1994))** The $\mathcal{L}_2$ gain of system (8) (which also equals to the $\mathcal{H}_\infty$ norm of the system's transfer matrix) is defined as the quantity

$$\sup_{\|\omega\|\neq 0} \frac{\|z\|_2}{\|w\|_2}, \tag{10}$$

where the supremum is taken over all nonzero trajectories of system (8), starting from $e(0) = 0$.

Based on the foregoing, the following problem is studied in this paper.

Problem 2.3. For system (8) design an output-feedback controller, with gain matrix $K \in \mathbb{R}^{m\times n}$, defined as

$$u = Ky - F(q_s, q_m), \tag{11}$$

such that the closed-loop system is asymptotically stable, and the $\mathcal{H}_\infty$ norm of the system's transfer matrix is strictly less than γ, where $\gamma \geq 0$ is a known constant defined by the designer.

Substituting the control law (11) to the system (8) one can obtain

$$\begin{aligned} \dot{e} &= Ae + B_1 w + Bu_1, \\ z &= C_1 e + D_{11} w + D_{12} u_1, \\ y &= Ce + D_{21} w, \end{aligned} \tag{12}$$

where

$$u_1 = Ky = KCe + KD_{12} w. \tag{13}$$

Remark 2.4. Control law (13) is defined in a static output-feedback (SOF) form. Let's remark that many controller structures can be transformed to the form (13) (like PI, PID, PD, even full/reduced order dynamic output-feedback controllers), by augmenting the system with additional state variables. For more info, see Ilka (2018) or Veselý and Rosinová (2013).

The following lemma (Isidori, 2011, 2017) plays an important role in the next development.

Lemma 2.5. Let $\gamma \geq 0$ be a known constant scalar. If there exists a controller gain matrix K and a positive definite matrix X satisfying

$$\begin{bmatrix} A_f^T X + XA_f, & XB_f, & C_f^T \\ Bf^T X, & -\gamma I, & D_f \\ C_f, & D_f, & -\gamma I \end{bmatrix} \prec 0, \tag{14}$$

$$A_f = A - BKC, \tag{15}$$

$$B_f = B_1 - BKD_{21}, \tag{16}$$

$$C_f = C_1 - D_{12}KC, \tag{17}$$

$$D_f = D_{11} - D_{12}KD_{21}, \tag{18}$$

then the closed-loop system formed by system (12) and control law (13) is asymptotically stable and the $\mathcal{H}_\infty$ norm of its closed-loop transfer function is strictly less than γ.

Proof. For proof see (Isidori, 2011, 2017). □

The Inequality (14) in Lemma 2.5 is known as the bounded real lemma (Isidori, 2011, 2017; Krokavec and Filasova, 2016). The problem with the application of the above inequality to design the robust $\mathcal{H}_\infty$ controller is that it gives rather conservative results. In this paper we propose a new robust $\mathcal{H}_\infty$ controller design procedure, using modified bounded real lemma and feedback linearization, to control the chaotic system (1) with less conservativeness.

3 Robust $\mathcal{H}_\infty$ontroller Design

This section formulates the theoretical approach to robust controller design with less conservative results.

Theorem 3.1. *Chaotic system (8) with any initial conditions can be stabilized by control algorithm (11) with $\mathcal{H}_\infty$ norm of the system's transfer matrix strictly less than γ, if there exists a symmetric positive definite matrix P, matrices $N_i, i = 1, 2, 3$ and controller gain matrix K such that the following inequality holds*

$$W = \begin{bmatrix} W_{11}, & W_{12}, & W_{13} \\ W_{12}^T, & W_{22}, & W_{23} \\ W_{13}^T, & W_{23}^T, & W_{33} \end{bmatrix} \prec 0, \tag{19}$$

where

$$\begin{aligned} W_{11} &= N_1 + N_1^T, \\ W_{12} &= -N_1^T A_f + N2 + P, \\ W_{13} &= -N_1^T B_f + N_3, \\ W_{22} &= -N_2^T A_f - A_f^T N_2 + C_f^T * C_f, \\ W_{23} &= -N_2^T B_f - A_f^T N_3 + C_f^T * D_f, \\ W_{33} &= -N_3^T B_f - B_f^T N_3 - \gamma^2 I_w + D_f^T * D_f, \end{aligned}$$

wherein A_f, B_f, C_f and D_f are defined in (15), (16), (17), and (18).

Proof. Suppose that there exists a quadratic function $V(e) = e^T Pe$, $P \succ 0$, and some $\gamma \geq 0$ such that:

$$\dot{V}(e) + z^T z - \gamma w^T w < 0, \tag{20}$$

Integrating (20) from 0 to T, with initial condition $e(0) = 0$, we can get:

$$V(e(T)) + \int_0^T (z^T z - \gamma^2 w^T w)dt < 0. \tag{21}$$

Since P is positive definite then $V(e(T)) > 0$, which implies

$$\frac{\|z\|_2}{\|w\|_2} \leq \gamma. \tag{22}$$

Furthermore, the inequality (22) and the Definition 2.2 implies that the $\mathcal{L}_2$ gain of the system (12) (which also equals to the $\mathcal{H}_\infty$ norm of the system's transfer matrix (Boyd et al., 1994)), is strictly less than γ.

By using the auxiliary matrices N_1, $N_2 \in \mathbb{R}^{n\times n}$ and $N_3 \in \mathbb{R}^{n\times k}$, and by substituting the control law (13) to the system (12), we can get:

$$H = 2(\dot{e}^T N_1^T + e^T N_2^T + \omega^T N_3^T)\\ (\dot{e} - A_f e - B_f w) = 0 \quad (23)$$

where A_f and B_f are defined in (15) and (16). Since $H = 0$, we can write

$$\dot{V}(e) + H + z^T z - \gamma\omega^T\omega = d^T W d < 0, \quad (24)$$

where for $d^T = [\dot{e}^T,\, e^T,\, w^T]$,

$$\dot{V}(e) = \dot{e}^T P e + e^T P \dot{e} = d^T \begin{bmatrix} 0, & P, & 0 \\ P, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix} d, \quad (25)$$

$$z^T z = d^T \begin{bmatrix} 0, & 0, & 0 \\ 0, & C_f^T C_f, & C_f^T D_f \\ 0, & D_f^T C_f, & D_f^T D_f \end{bmatrix} d, \quad (26)$$

$$\gamma^2 \omega^T \omega = d^T \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & \gamma^2 I_{n_\omega} \end{bmatrix} d, \quad (27)$$

$$H = d^T \begin{pmatrix} N_1 + N_1^T, & -N_1^T A_f + N2, & -N_1^T B_f + N_3 \\ *, & -N_2^T A_f - A_f^T N_2, & -N_2^T B_f - A_f^T N_3 \\ *, & *, & -N_3^T B_f - B_f^T N_3 \end{pmatrix} d, \quad (28)$$

wherein, C_f and D_f are defined in (17) and (18). Finally, from (24) we can get (19), which completes the proof. □

Theorem 3.1 is formulated as a feasibility problem, for given known γ. The next Corollary completes it with minimization of γ to obtain minimal $\mathcal{H}_\infty$ norm.

Corollary 3.2. *If the following optimization problem has a solution, then the closed-loop system formed by system (8) and controller (11) will be stable with minimal $\mathcal{H}_\infty$ norm*

$$\min_{F,P,N_1,N_2,N_3,\gamma} (\gamma) \quad (29)$$

s.t.:

$$W \prec 0, \quad (30)$$

$$P \succ 0, \quad (31)$$

$$\gamma \geq 0. \quad (32)$$

4 Examples

Example 4.1. In order to evaluate the conservativeness of the previous proposed method (Corollary 3.2) the COMP$l_e$ib library (Leibfritz, 2004) has been used (Table 1). In order to better highlight the advantages of the proposed method beside approaches from (Hoi et al., 2003, Problem 2) and from (Isidori, 2017, Theorem 3.1), which are also based on bilinear matrix inequalities, we have decided to include results from other approaches like the HIFOO toolbox (Burke et al., 2006) and the BMISolver toolbox (Dinh et al., 2011a) as well. The HIFOO toolbox is based on a hybrid algorithm for nonsmooth, nonconvex optimization, which uses several techniques, namely quasi-Newton updating, bundling and gradient sampling. The BMISolver toolbox is combining convex-concave decompositions and linearization approaches for solving BMIs.

Numerical solutions for the proposed method (Corollary 3.2) have been carried out by PENBMI 2.1 solver (Henrion et al., 2005) under Matlab R2018a (The Mathworks, Inc., 2018) using YALMIP (Löfberg, 2004) on NEOS Server Version 5.0 (Czyzyk et al., 1998). Numerical solutions for HIFOO and BMISolver Toolboxes as well as for the BMI formulation based on (Hoi et al., 2003, Problem 2) have been taken over from (Dinh et al., 2011b, Table 2). Finally, numerical solutions for the BMI formulation based on (Isidori, 2017, Theorem 3.1) have been carried out by PENLAB solver (Fiala et al., 2013) under Matlab R2017a using YALMIP on HP EliteBook 820 notebook. Table 1 indicates that the proposed approach (Corollary 3.2) is less conservative compared to other approaches using BMIs ((Hoi et al., 2003, Problem 2) and (Isidori, 2017, Theorem 3.1)) since it can handle much more examples from the COMP$l_e$ib library. Table 1 also proves that the proposed approach with the used relaxation (slack variable approach) combined with PENBMI global solver outperforms the majority of the algorithms in most cases. In addition, thanks to the reduced conservativeness, it can handle much more examples from the COMP$l_e$ib library compared to other BMI-based approaches, which indicates that the proposed approach could be an efficient and reliable computer-aided control system design tool for small and medium sized problems, with a potential for realistic industrial applications as well.

Example 4.2. Numerical simulation of the well known Van der Pol oscillator model has been used to verify the results of the Theorem 3.1 in the second example of this article. The master Van der Pol chaotic system can be described as

$$\dot{q}_{m_1} = q_{m_2} \quad (33)$$

$$\dot{q}_{m_2} = -q_{m_1} + (1 - \epsilon_m q_{m_1}^2) q_{m_2}, \quad (34)$$

Table 1: $\mathcal{H}_\infty$ benchmarks on COMP$l_e$ib plants

Problem description						**γ - Other methods**				**Proposed method**		
Name	n_x	n_y	n_u	n_z	n_ω	Burke et al. (2006) (HIFOO toolbox)	Dinh et al. (2011a) (BMISolver toolbox)	Hoi et al. (2003)* (BMI)	Isidori (2017)* (BMI)	γ	iter.	time (s)
AC6	7	4	2	7	7	4.1140	4.1954	-	-	**4.1140**	21	4.72371
AC7	9	2	1	1	4	0.0651	0.0339	0.3810	0.0651	**0.0599**	235	330.525
AC8	9	5	1	2	10	2.0050	4.5463	-	-	**2.0050**	30	23.4929
AC17	4	2	1	4	4	6.6124	6.6571	-	6.7705	**6.6124**	56	3.59318
HE3	8	6	4	10	1	0.8545	0.8640	1.6843	1.0016	**1.000**	25	48.5127
REA3	12	1	3	12	12	74.2513	75.0634	74.4460	-	**74.2513**	14	84.2832
DIS2	3	2	2	3	3	1.0548	1.1570	-	-	**1.0231**	202	6.70886
BDT1	11	3	3	6	1	0.2664	0.8544	-	0.2667	**0.2662**	70	719.727
CSE1	20	10	2	12	1	0.0201	0.0219	-	-	**0.0200**	86	215.603
EB1	10	1	1	2	2	3.1225	2.0532	39.9526	3.1225	**1.8979**	29	51.5588
EB2	10	1	1	2	2	2.0201	0.8150	39.9547	2.0201	**0.8142**	20	18.6961
EB3	10	1	1	2	2	2.0575	0.8157	3995311.0743	2.0575	**0.8143**	179	367.619
TF1	7	2	4	4	1	0.3710	-	-	-	**2.0013**	60	29.8262
PSM	7	3	2	5	2	0.9202	0.9266	-	0.9202	**0.9202**	252	118.164
NN2	2	1	1	2	2	2.2216	2.2216	-	2.2216	**2.2216**	15	0.00839
NN4	4	3	2	4	4	1.3627	1.3884	-	-	**1.3587**	80	9.42532
NN15	3	2	2	4	1	0.1039	0.1201	-	0.0985	**0.0981**	81	1.28827

and the slave system as

$$\dot{q}_{s_1} = q_{s_2}, \tag{35}$$

$$\dot{q}_{s_2} = -q_{s_1} + (1 - \epsilon_s q_{s_1}^2) q_{s_2} + u, \tag{36}$$

wherein $\epsilon_m > 0$ and $\epsilon_s > 0$. By performing the operation (7) the model for tracking error can be obtained as

$$\dot{e}_1 = e_2, \tag{37}$$

$$\dot{e}_2 = -e_1 + e_2 - \epsilon_s q_{s_1}^2 q_{s_2} + \epsilon_m q_{m_1}^2 q_{m_2} + u, \tag{38}$$

which can be transformed to the form (8) with

$$A = \begin{bmatrix} 0, & 1 \\ -1, & 1 \end{bmatrix},\ B = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = [1,\ 1],$$

$$C = I_n,\ D_{11} = D_{12} = D_{21} = 0,$$

$$F(q_s, q_m) = \begin{bmatrix} 0 \\ -\epsilon_s q_{s_1}^2 q_{s_2} + \epsilon_m q_{m_1}^2 q_{m_2} \end{bmatrix}.$$

Note that matrix A is unstable. The control law (13) is described by the following equation

$$u = Ky - F(q_s, q_m) = Ke - \epsilon_s q_{s_1}^2 q_{s_2} + \epsilon_m q_{m_1}^2 q_{m_2},$$

where the obtained gain matrix K (using Theorem 3.1), for different initial parameters are

- for $0 < P < \rho I$, $\rho = 100$, and $\gamma = 0.4151$

$$K = [-2.8726,\ -5.7697]. \tag{39}$$

  The maximal real part of the eigenvalues of the closed-loop system is $\lambda_m = -1.0377$.

- for $\rho = 500$ and $\gamma = 0.326$

$$K = [-8.7081,\ -13.5514]. \tag{40}$$

  The maximal real part of the eigenvalues of the closed-loop system is $\lambda_m = -0.8281$.

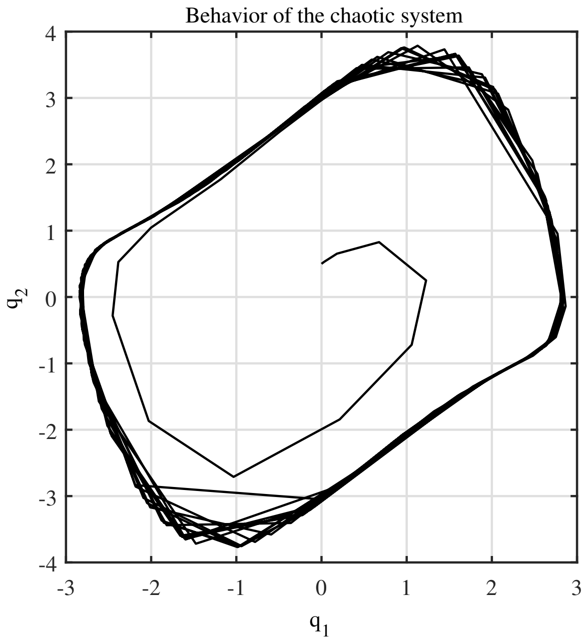

Figure 1: Behaviour of the first chaotic system.

Note that minus sign of matrix K provides negative feedback.

The behaviour of the chaotic system for non-zero initial conditions is shown in Fig. 1. The time responses of the tracking error are shown in Fig. 2 (for (39)) and in Fig. 4 (for (40)). The measurements of γ according to ω (rad s^{-1}) are shown in Fig. 3 (for (39)) and in Fig. 5 (for (40)). In both cases the measured value γ is less than the defined maximum value.

Example 4.3. The third example is borrowed from Wang et al. (2012), where the third order master

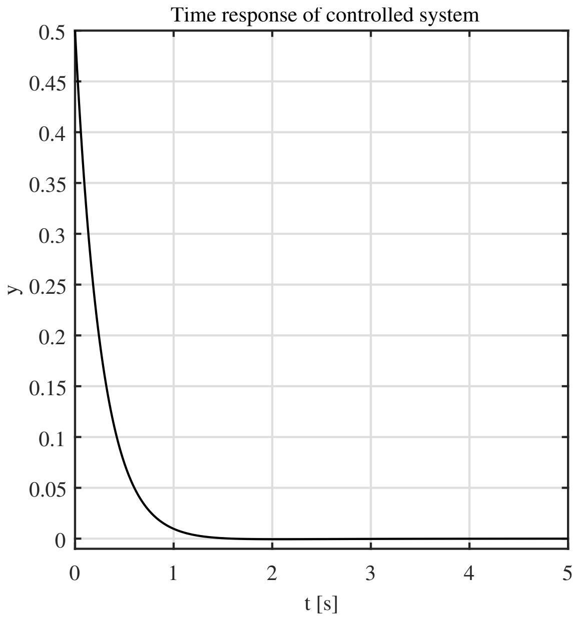

Figure 2: Time response of the controlled output variable.

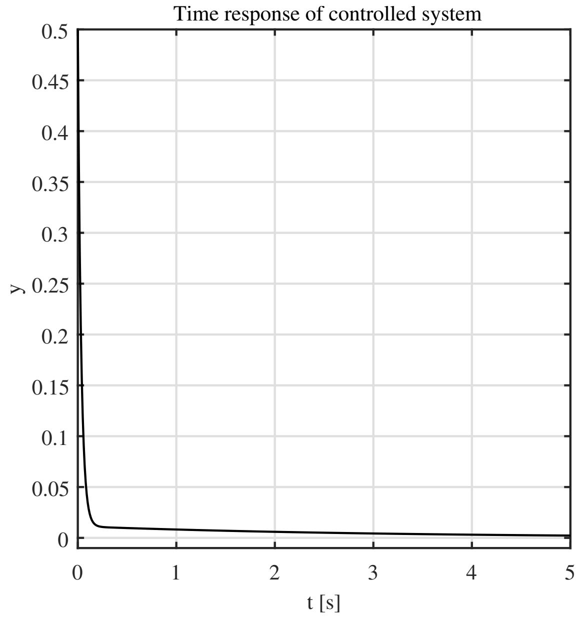

Figure 4: Time response of the controlled output variable.

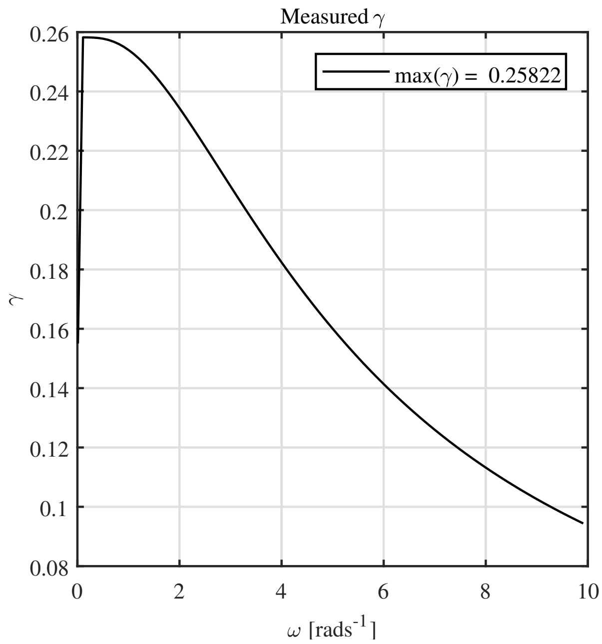

Figure 3: Measured γ according to ω.

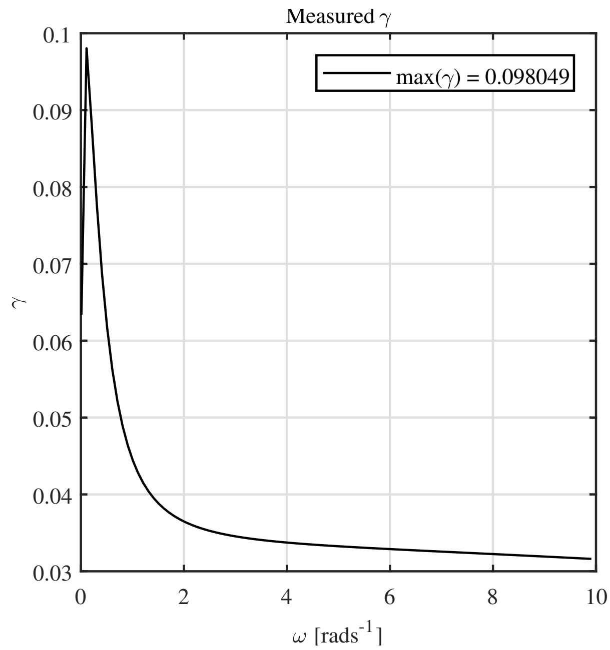

Figure 5: Measured γ according to ω.

chaotic system is defined as

$$\dot{q}_{m_1} = q_{m_2}, \quad (41)$$
$$\dot{q}_{m_2} = q_{m_3}, \quad (42)$$
$$\dot{q}_{m_3} = 5.5q_{m_1} - 3.5q_{m_2} - q_{m_3} + q_{m_1}^3, \quad (43)$$

and the slave system as

$$\dot{q}_{s_1} = q_{s_2}, \quad (44)$$
$$\dot{q}_{s_2} = q_{s_3}, \quad (45)$$
$$\dot{q}_{s_3} = -1.2q_{s_1} - q_{s_2} - 0.6q_{s_3} + q_{s_1}^2 + w + u, \quad (46)$$

By performing the operation (7) the model for tracking error can be obtained in the form (8) with

$$A = \begin{bmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ -1.2, & -1, & -0.6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = [1.1,\ 0.6,\ 0.7],\ C = I_n,\ D_{11} = D_{12} = D_{21} = 0,$$

$$F(q_s, q_m) = \begin{bmatrix} 0 \\ 0 \\ f(q_s, q_m) \end{bmatrix},$$

$$f(q_s, q_m) = -6.7q_{m_1} + 2.5q_{m_2} + 0.4q_{m_3} + q_{s_1}^2 - q_{m_1}^3.$$

The obtained gain matrix K using Theorem 3.1 for $\gamma = 0.2908$, and $\rho = 50$ is

$$K = [-4.9405,\ -5.5067,\ -5.383]. \quad (47)$$

The maximal real part of the eigenvalues of the closed-loop system is $\lambda_m = -0.535$. Dynamic behaviour of the tracking error is shown in Fig. 6, and the measurement of γ according to ω is shown in Fig. 7.

Example 4.4. The fourth example has been borrowed from Wang et al. (2012), where the second order master chaotic system is defined as

$$\dot{q}_{m_1} = q_{m_2}, \quad (48)$$
$$\dot{q}_{m_2} = -0.4q_{m_2} + 1.1q_{m_1} - q_{m_1}^3 - 2.1cos(1.8t), \quad (49)$$

and the slave system as

$$\dot{q}_{s_1} = q_{s_2}, \quad (50)$$
$$\dot{q}_{s_2} = q_{s_1} - 0.5q_{s_2} - 0.8q_{s_1}^3 - 2cos(1.5t) + w + u, \quad (51)$$

By performing the operation (7) the model for tracking error can be obtained in the form (8) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

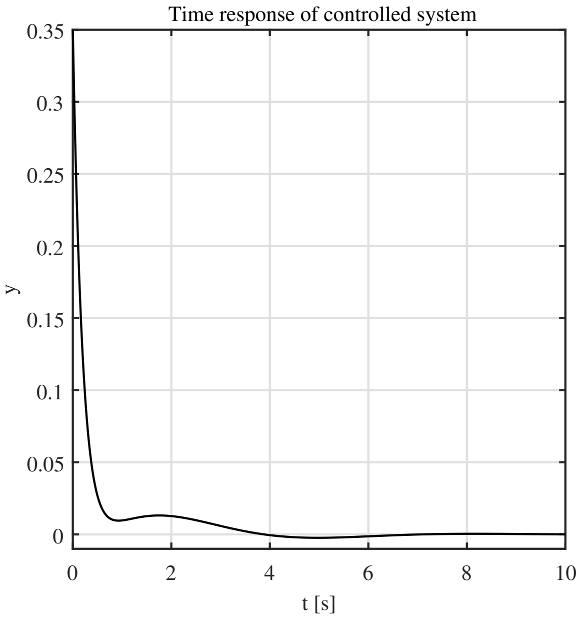

Figure 6: Time response of the controlled output variable.

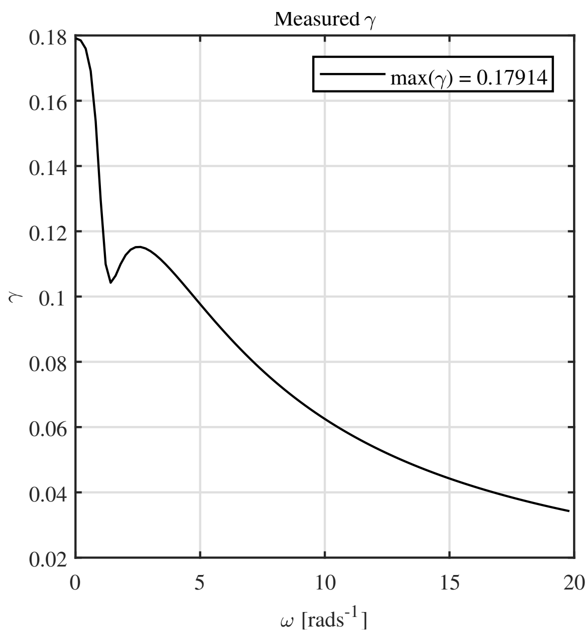

Figure 7: Measured γ according to ω.

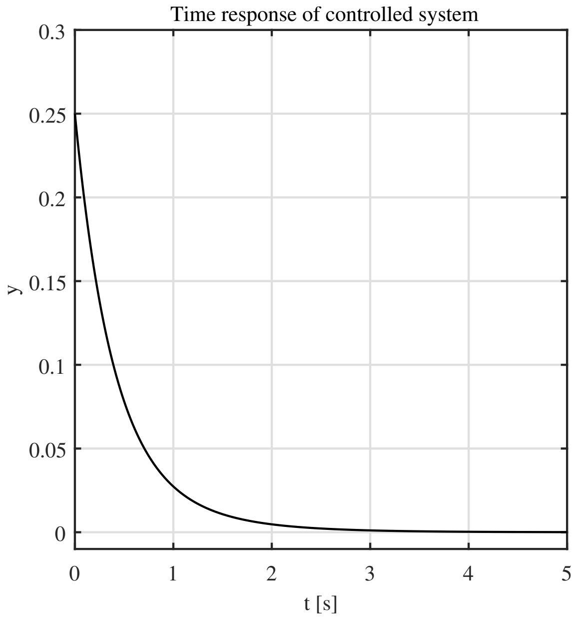

Figure 8: Time response of the controlled output variable.

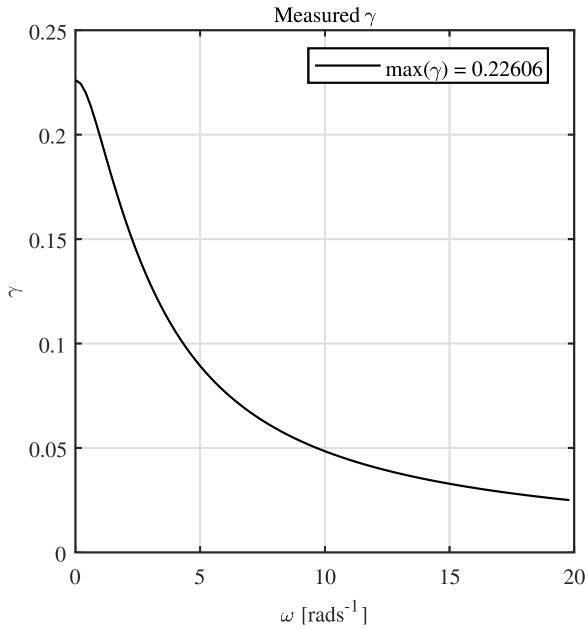

Figure 9: Measured γ according to ω.

$$
\begin{aligned}
C_1 = [0.8,\ 0.5],\ C = I_n,\ D_{11} = D_{12} = D_{21} = 0,\\
F(q_s, q_m) = \begin{bmatrix} 0 \\ f(q_s, q_m, t) \end{bmatrix},\\
f(q_s, q_m, t) = &-0.1q_{m_2} + q_{s_1} - 0.8q_{s_1}^3 - 2cos(1.5t)\\
&- 1.1q_{m_1} + q_{m_1}^3 + 2.1cos(1.8t).
\end{aligned}
$$

The obtained control algorithm (using Theorem 3.1) is

$$
\begin{aligned}
u =&Kx + 0.1q_{m_2} - q_{s_1} + 0.8q_{s_1}^3 + 2cos(1.5t)\\
&+ 1.1q_{m_1} - q_{m_1}^3 - 2.1cos(1.8t),
\end{aligned}
\tag{52}
$$

where the gain matrix

$$
K = [-3.5389,\ -3.5174], \tag{53}
$$

for $\gamma = 0.3497$ and $\rho = 50$. The maximal real part of the eigenvalues of the closed-loop system is $\lambda_m = -1.3044$. Dynamic behaviour of the tracking error is shown in Fig. 8, and the measurement of γ according to ω in Fig. 9.

In all cases the tracking error converged to zero (Fig. 2, Fig. 4, Fig. 6, and Fig. 8) which means that the controlled slave system tracked the master system. Also in all cases the measured γ was lower than the numerically computed maximum.

5 Conclusion

In this paper a new bounded real lemma is proposed for robust master-slave controller design that is used for synchronization of generalized chaotic systems. The modified $\mathcal{L}_2$ gain approach and the new bounded real lemma ensures less conservative BMI-based controller design due to the introduced auxiliary matrices. When the robust stability conditions hold the closed-loop system is asymptotically stable and the $\mathcal{H}_\infty$ norm of closed-loop transfer function with respect to the defined output and input is strictly less than $\gamma \geq 0$. Numerical examples show the effectiveness of the proposed method.

Acknowledgments

The work was supported by Grant 1/0475/16 of the Slovak Grant Agency, and by the Swedish Energy Agency (P43322-1).

References

Boccaletti, S., Kurths, J., Osipov, G., Valladares, D., and Zhou, C. The synchronization of chaotic systems. *Physics Reports*, 2002. 366(1):1 – 101. doi:10.1016/S0370-1573(02)00137-0.

Boyd, S., Ghaoui, L. E., Feron, E., and Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory*. Number 15 in Studies in applied mathematics. SIAM, 1994.

Burke, J. V., Henrion, D., Lewis, A. S., , and Overton, M. L. HIFOO - a MATLAB Package for Fixed-order

Controller Design and H-infinity Optimization. In *IFAC Symposium on Robust Control Design*. pages 339–344, 2006.

Chen, H.-H. Chaos control and global synchronization of liu chaotic systems using linear balanced feedback control. *Chaos, Solitons & Fractals*, 2009. 40(1):466 – 473. doi:10.1016/j.chaos.2007.07.098.

Czyzyk, J., Mesnier, M. P., and Moré, J. J. The neos server. *IEEE Journal on Computational Science and Engineering*, 1998. 5(3):68–75.

Dinh, Q. T., Gumussoy, S., Michiels, W., and Diehl, M. Combining convex-concave decompositions and linearization approaches for solving bmis, with application to static output feedback. Technical report, Department of Electrical Engineering (ESAT-SCD) and Optimization in Engineering Center (OPTEC), K.U. Leuven, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium, 2011a. URL `http://arxiv.org/abs/1202.5488`.

Dinh, Q. T., Michiels, W., and Diehl, M. An inner convex approximation algorithm for bmi optimization and applications in control. Technical report, Department of Electrical Engineering (ESAT-SCD) and Optimization in Engineering Center (OPTEC), K.U. Leuven, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium, 2011b. URL `http://arxiv.org/abs/1202.5488`.

Farghaly, A. A. M. Chaos synchronization of complex rössler system. *Applied Mathematics & Information Sciences*, 2013. 7(4):1415–1420.

Fiala, J., Kočvara, M., and Stingl, M. Penlab: A matlab solver for nonlinear semidefinite optimization, 2013. Submitted to Mathematical Programming Computation.

Henrion, D., Löfberg, J., Kočvara, M., and Stingl, M. Solving polynomial static output feedback problems with PENBMI. In *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05*. pages 7581–7586, 2005.

Hoi, C., Scherer, C., van der Mech, E., and Bosgra, O. A nonlinear sdp approach to fixed-order controller synthesis and comparison with two other methods applied to an active suspension system*. *European Journal of Control*, 2003. 9(1):13 – 28. doi:10.3166/ejc.9.13-28.

Ilka, A. Matlab/Octave toolbox for structurable and robust output-feedback LQR design. *IFAC-PapersOnLine*, 2018. 51(4):598 – 603. doi:10.1016/j.ifacol.2018.06.161. 3rd IFAC Conference on Advances in Proportional-Integral-Derivative Control PID 2018.

Isidori, A. Robust stability via H_∞ methods. Technical report, European Embedded Control Institute, 2011. URL `http://www.eeci-institute.eu/GSC2012/Photos-EECI/EECI-GSC-2012-M9/Handout_1.pdf`.

Isidori, A. *Lectures in Feedback Design for Multivariable Systems*. Springer International Publishing, 1st edition, 2017. doi:10.1007/978-3-319-42031-8.

Krokavec, D. and Filasova, A. On enhanced mixed H_2/H_∞; design conditions for control of linear time-invariant systems. In *2016 17th International Carpathian Control Conference (ICCC)*. pages 384–389, 2016. doi:10.1109/CarpathianCC.2016.7501128.

L Fradkov, A. *Cybernetical Physics - From Control of Chaos to Quantum Control*. Springer, 2007.

Leibfritz, F. COMP$l_e$ib: COnstraint Matrix-optimization Problem library - a collection of test examples for nonlinear semidefinite programs, control system design and related problems. Technical report, University of Trier, Department of Mathematics, D-54286 Trier, Germany, 2004. URL `http://www.friedemann-leibfritz.de/COMPlib_Data/COMPlib_Main_Paper.pdf`.

Löfberg, J. YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In *Proceedings of the CACSD Conference*. Taipei, Taiwan, pages 284–289, 2004.

Mobayen, S. and Javadi, S. Disturbance observer and finite-time tracker design of disturbed third-order nonholonomic systems using terminal sliding mode. *Journal of Vibration and Control*, 2015. doi:10.1177/1077546315576611.

Mobayen, S. and Tchier, F. Synchronization of a class of uncertain chaotic systems with lipschitz nonlinearities using state-feedback control design: A matrix inequality approach. *Asian Journal of Control*, 2017. 20. doi:10.1002/asjc.1512.

Pecora, L. and Carroll, T. Synchronization in chaotic system. *Physical Review Letters*, 1990. 64:821.

Shahverdiev, E., Hashimova, L., and Hashimova, N. Chaos synchronization in some power systems. *Chaos, Solitons & Fractals*, 2008. 37(3):827 – 834. doi:10.1016/j.chaos.2006.09.071.

Song, Y. and Yu, X. Chaos synchronization from an invariant manifold approach. *Dynamics of Continuous, Discrete and Impulsive Systems. Series B: Applications and Algorithms*, 2003. 10(6):879–890.

Sprott, J. C. New chaotic regimes in the lorenz and chen systems. *International Journal of Bifurcation and Chaos*, 2015. 25:1550033.

The Mathworks, Inc. *MATLAB R2018a*. The Mathworks, Inc., Natick, Massachusetts, 2018.

Veselý, V. and Rosinová, D. Robust pid-psd controller design: Bmi approach. *Asian Journal of Control*, 2013. 15(2):469–478. doi:10.1002/asjc.559.

Wang, B., Shi, P., Reza Karimi, H., and Wang, J. H_∞ robust controller design for the synchronization of master-slave chaotic systems with disturbance input. *Modeling, Identification and Control: A Norwegian Research Bulletin*, 2012. 33. doi:10.4173/mic.2012.1.3.

Wang, F. and Balakrishnan, V. Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems. *IEEE Transactions on Automatic Control*, 2002. 47(5):720–734. doi:10.1109/TAC.2002.1000267.

Zeng, H.-B., Park, J. H., Xiao, S.-P., and Liu, Y. Further results on sampled-data control for master–slave synchronization of chaotic lur'e systems with time delay. *Nonlinear Dynamics*, 2015. 82(1):851–863. doi:10.1007/s11071-015-2199-6.