



# Stochastic Stability Analysis for Markovian Jump Neutral Nonlinear Systems

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## Abstract

In this paper, the stability problem is studied for a class of Markovian jump neutral nonlinear systems with time-varying delay. By Lyapunov-Krasovskii function approach, a novel mean-square exponential stability criterion is derived for the situations that the systems transition rates are completely accessible, partially accessible and non-accessible, respectively. Moreover, the developed stability criterion is extended to the systems with different bounded sector nonlinear constraints. Finally, some numerical examples are provided to illustrate the effectiveness of the proposed methods.

*Keywords:* Stochastic stability; Markovian jump system, nonlinear system, time-varying delay

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## 1 Introduction

Among stochastic hybrid systems, a widely studied class is the so-called Markovian jump systems (MJSs), which are described by a set of systems with commutation generated by a finite state Markov chain. It has been shown that MJSs have great application potentials, such as biomedicine, fault tolerant systems, networked control, economics, aerospace, and communication networks. Due to the probabilistic description of commutations, MJSs are very suitable and effective to describe dynamical systems with the model changes induced by external causes, such as unexpected events, random faults, uncontrolled configuration changes. A great amount of effort has been devoted to MJSs in the past years, and many useful and important results have been reported in literature. In [Ma et al. \(2012\)](#),

a stochastic version of a bounded real lemma was derived for a general class of discrete-time time-varying MJSs with state- and disturbance-multiplicative noise, and the necessary and sufficient conditions for the solvability of the control problem in terms of four coupled discrete-time Riccati equations are presented. In [Zhang \(2009\)](#), the estimation problem was studied for a class of MJSs with time-varying transition probabilities, which was considered to be finite piecewise homogeneous. In [Costa and Oliveira \(2012\)](#), the stochastic optimal control problem was researched for discrete-time linear systems subject to Markov jumps and multiplicative noises under two criteria. In [Fei et al. \(2009\)](#), a new delay-dependent stochastic stability criterion for a class of MJSs with time delay was derived based on a novel Lyapunov-Krasovskii functional ap-

proach. In Wu et al. (2012), the problem on the sliding mode control of Markov jump singular time-delay systems is concerned through investigation of the bounded gain performance in the analysis of sliding mode dynamics in order to improve the system transient performance. Recently in Karimi (2011), an efficient approach is developed for robust  $H_\infty$  control problem of uncertain time-delay systems with Markovian switching parameters and mixed discrete, neutral and distributed delays. For more state-of-the-art results on this topic, readers are referred to (Karimi (2012), Wu et al. (2011), Basin and Shi (2009), Shi and Liu (2011), Wu et al. (2010), Liu et al. (2011b)) and the references therein.

In real systems, time delays exist widely. Hence recently much effort has been included in such area, see Cai et al. (2012), Liu et al. (2011a) and references therein. Meanwhile, the transition rates of MJSs are, in general, not completely accessible. Hence the problems of modeling, analysis and design on MJSs with partially transition rates have become an important and challenging issue, and attracted a lot of attentions, see for example, Shi et al. (2012), Yin et al. (2012), Luan et al. (2010), Zhang et al. (2009), Zhang et al. (2011), Zhang et al. (2009), Wang et al. (2010) and the references therein. However, most theoretical results either have some strict limitations with regard to transition rates matrix, such as main diagonal elements are required, or have some conservatism for not fully utilizing the provided transition rates. In addition, for MJSs with time-varying delay, most stability criteria have the constraints that the derivative of time-varying delay must be less than a constant, such as 1, which limits the applied scope. In this paper, all the available commutation information will be utilized to derive a stochastic stability criterion, independent of time-varying delays derivative, for Markovian jump neutral nonlinear delay systems (MJNNSs), no matter if transition rates are completely accessible, partially accessible and non-accessible. Moreover, the developed stability criterion will be extended to systems with different bounded sector nonlinear constraints. Some numerical examples will be included to demonstrate the effectiveness of the obtained theoretical results.

The remainder of this paper is organized as follows. The MJNNS with time-varying delay and some preliminaries are provided in Section 2. Section 3 presents the results on stochastic stability analysis. Section 4 gives an illustrative example, and the paper is concluded in Section 5. Notations used in this paper are fairly standard. Let  $R^n$  be the  $n$ -dimensional Euclidean space,  $R^{n \times m}$  represents the set of  $n \times m$  real matrices, the symbol  $*$  denotes the elements below the main diagonal of a symmetric block matrix,  $A > 0$  means that

$A$  is a real symmetric positive definitive matrix,  $I$  denotes the identity matrix with appropriate dimensions.  $diag\{\dots\}$  denotes the diagonal matrix.  $E\{\cdot\}$  refers to the expectation operator with respect to some probability measure  $P$ .  $\|\cdot\|$  refers to the Euclidean vector norm or the induced matrix 2-norm. The superscript  $T$  stands for matrix transposition,  $L_{n,h} = L([-h, 0], R^n)$  denotes the Banach space of continuous functions mapping the interval  $[-h, 0]$  into  $R^n$  with the topology of uniform convergence.

## 2 Model description and preliminaries

In this paper,  $(\Omega, \Upsilon, P)$  denotes the probability space, where  $\Omega$  is the sample space,  $\Upsilon$  is  $\sigma$ -algebra of subset of the sample space, and  $P$  is the probability measure defined on  $\Upsilon$ . Let  $S = \{1, 2, \dots, N\}$  and the random process  $\{r(t), t \in [0, +\infty)\}$  be a homogeneous, finite state Markov process with right continuous trajectories with generator  $\Pi = (\pi_{ij})_{N \times N}$  and transition probability form mode  $i$  at time  $t$  to mode  $j$  at time  $t + \Delta t, i \in S, j \in S$ , given by

$$P\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & j \neq i \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & j = i \end{cases}$$

where transition rates  $\pi_{ij} \geq 0, j \neq i, \pi_{ii} = -\sum_{j=1, j \neq i}^N \pi_{ij}, \Delta t > 0$ , and  $\lim_{\Delta t \rightarrow 0} o(\Delta t)/\Delta t = 0$ .

For the convenience of study, the transition rates of the Markov chain are considered to be partially accessible. Namely, some elements in transition rates matrix are time invariant but unknown, such as a system with three operation modes may have the transition rates matrix as follow

$$\Pi = \begin{bmatrix} - & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ - & \pi_{32} & - \end{bmatrix} < 0$$

where  $-$  represents the inaccessible element.

Throughout the paper, we denote  $S = S_{kn}^i \cup S_{un}^i$  for  $\forall i \in S$ , where  $S_{kn}^i = \{j : \pi_{ij} \text{ is known}\}$  and  $S_{un}^i = \{j : \pi_{ij} \text{ is unknown}\}$ . First, consider the following MJNNS

$$\begin{cases} \dot{x}(t) = A(r(t))x(t) + B(r(t))x(t - \tau(t)) \\ \quad + C(r(t))\dot{x}(t - d) + D(r(t))f(t, z(t)) \\ z(t) = Lx(t) + Nx(t - \tau(t)) \\ x(t_0 + \theta) = \psi(\theta), \theta \in [-h, 0] \end{cases} \quad (1)$$

where  $x(t) \in R^n$  is the state vector,  $0 \leq d \leq h$  is the neutral delay,  $0 \leq \tau(t) \leq h$  is the time-varying

delay,  $A(r(t)) \in R^{n \times n}, B(r(t)) \in R^{n \times n}, C(r(t)) \in R^{n \times n}, D(r(t)) \in R^{n \times n}, L \in R^{m \times n}, N \in R^{m \times n}$  are matrix functions of the random jumping process  $\{r(t)\}$ . When  $r(t) = i \in S$ , the matrices  $A(r(t)), B(r(t)), C(r(t)), D(r(t))$  are represented by  $A_i, B_i, C_i, D_i$ ,  $\psi(\theta) \in L_{n,h}$  is the initial condition,  $f(t) \in R^n$  is the nonlinear term, and satisfies the following Lipchitz condition

$$f^T(t, z(t)) (f(t, z(t)) - Kz(t)) \leq 0$$

We have

$$-2f^T(t, z(t))f(t, z(t)) + 2f^T(t, z(t))K(Lx(t) + Nx(t - \tau(t))) \leq 0$$

Then, model (1) can be represented as

$$\begin{cases} \dot{x}(t) = y(t) \\ y(t) = A(r(t))x(t) + B(r(t))x(t - \tau(t)) \\ \quad + C(r(t))y(t - d) + D(r(t))f(t, z(t)) \\ z(t) = Lx(t) + Nx(t - \tau(t)) \end{cases}$$

**Remark 1:** It is shown in [Bao et al. \(2009\)](#) that the  $L([-h, 0], R^n) \times S$  - valued process  $(x_t, r(t))$  is a time homogenous strong Markov process, where  $x_t = \{x(t + \theta) : -h \leq \theta \leq 0\}$ .

Now, to present the main objective of this paper more precisely, we also introduce the following definition for system (1), which is essential for the later development.

**Definition 1:** System (1) is mean-square exponentially stable if there exist scalars  $\alpha > 0$  and  $\beta > 0$  such that

$$E\|x(t)\|^2 \leq \alpha e^{-\beta t} \|\psi\|_h^2 \quad (2)$$

where  $\|\psi\|_h = \max \left\{ \sup_{h \leq \theta \leq 0} \|\psi(\theta)\| \right\}$ .

Therefore, in this paper, the authors attention will be focused on the development of mean-square exponential stability criterion independent of time-varying delays derivative for the situations that the systems transition rates are completely accessible, partially accessible and non-accessible, respectively.

Before ending the section, we give the following lemmas, which will be used in the proof of our main results.

**Lemma 1** ([Boyd et al. \(1994\)](#)): For any constant matrices  $E, G$  and  $F$  with appropriate dimensions,  $F^T F \leq kI$ ,  $k$  is a positive scalar, then

$$2x^T EFGy \leq cxEE^T x + \frac{k}{c} y^T G^T G y \quad (3)$$

where  $c$  is a positive scalar,  $x \in R^n$ , and  $y \in R^n$ .

**Lemma 2** ([Gu \(2000\)](#)): For any positive definite matrix  $\Phi \in R^{n \times n}$ , scalar  $\gamma > 0$ , vector function  $w :$

$[0, \gamma] \rightarrow R^n$  such that the integrations concerned are well defined, then

$$\left( \int_0^\gamma w(s) ds \right)^T \Phi \left( \int_0^\gamma w(s) ds \right) \leq \gamma \int_0^\gamma w^T(s) \Phi w(s) ds \quad (4)$$

### 3 Main results

In this section, based on Lyapunov method and linear matrix inequality techniques, the following stochastic stability criteria are derived.

**Theorem 1:** When  $\pi_{ii}$  is unknown with given  $h$  and  $k$ , if there exist  $P_2, P_3, U, V, W$ , and symmetric positive-definite matrices  $F_i, R, M, S, Q$ , such that

$$\begin{bmatrix} \left( \sum_{j \in S_{kn}^i} \pi_{ij} \right) T_i & T_{i1}^{kn} \\ * & T_{i2}^{kn} \end{bmatrix} < 0 \quad (5)$$

$$\begin{bmatrix} T_i & T_{i1}^{uk} \\ * & T_{i2}^{uk} \end{bmatrix} < 0, \forall j \neq i, j \in S_{uk}^i \quad (6)$$

$$\begin{bmatrix} W & U \\ * & S - R_{22} \end{bmatrix} > 0 \quad (7)$$

$$\begin{bmatrix} W & V \\ * & S \end{bmatrix} > 0 \quad (8)$$

where

$$\Sigma_i = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \Sigma_{1,3} & \Sigma_{1,4} & \Sigma_{1,5} & \Sigma_{1,6} \\ * & \Sigma_{2,2} & \Sigma_{2,3} & \Sigma_{2,4} & \Sigma_{2,5} & \Sigma_{2,6} \\ * & * & \Sigma_{3,3} & 0 & 0 & 0 \\ * & * & * & \Sigma_{4,4} & 0 & 0 \\ * & * & * & * & \Sigma_{5,5} & 0 \\ * & * & * & * & * & \Sigma_{6,6} \end{bmatrix}$$

$$\begin{aligned} \Sigma_{1,1} &= P_2 A_i + A_i^T P_2^T + Q + k F_i \\ \Sigma_{1,2} &= F_i - P_2 + A_i^T P_3^T, \Sigma_{1,3} = P_2 B_i + R_{12}^T \\ \Sigma_{2,2} &= h e^{kh} S - P_3 - P_3^T + M, \Sigma_{2,3} = P_3 B_i \\ \Sigma_{3,3} &= h R_{11} - R_{12} - R_{12}^T, \Sigma_{1,4} = P_2 C_i \\ \Sigma_{2,4} &= P_3 C_i, \Sigma_{4,4} = -M e^{-kh} \\ \Sigma_{1,5} &= P_2 D_i + L^T K^T, \Sigma_{2,5} = P_3 D_i \\ \Sigma_{3,5} &= N^T K^T, \Sigma_{5,5} = -2I, \Sigma_{6,6} = -Q e^{-kh} \\ T_i &= \Sigma_i + \Omega + \Omega^T + h e^{kh} W \\ T_{i1}^{kn} &= \left[ \sum_{j \in S_{kn}^i} \pi_{ij} F_j, 0, 0, 0, 0, 0 \right]^T \\ T_{i2}^{kn} &= - \sum_{j \in S_{kn}^i} \pi_{ij} F_j, T_{i1j}^{uk} = [F_j, 0, 0, 0, 0, 0]^T \\ T_{i2j}^{uk} &= -F_j, \Omega = [U, 0, -U + V, 0, 0, -V] \end{aligned}$$

then, MJNNS (1) is mean-square exponentially stable.

Proof: Choose the following Lyapunov-Krasovskii functional candidate

$$V(t, r(t)) = \sum_{j=1}^6 V_j(t, r(t)) \quad (9)$$

where

$$\begin{aligned} V_1(t, r(t)) &= \begin{pmatrix} x^T(t) & y^T(t) \end{pmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} F_i(r(t)) & 0 \\ P_2^T & P_3^T \end{bmatrix} \begin{pmatrix} x^T(t) & y^T(t) \end{pmatrix}^T \\ V_2(t, r(t)) &= \int_{-h}^0 \int_{t+\beta}^t y^T(\alpha) e^{k(\alpha-t+h)} S y(\alpha) d\alpha d\beta \\ V_3(t, r(t)) &= \int_{-h}^t \int_{\beta-\tau(\beta)}^{\beta} \eta^T e^{k(\beta-t)} R \eta d\alpha d\beta \\ V_4(t, r(t)) &= \int_{t-d}^t y^T(s) e^{k(s-t)} M y(s) ds \\ V_5(t, r(t)) &= \int_{-h}^0 \int_{t+\beta}^t \xi^T(\alpha) e^{k(\alpha-t+h)} W \xi(\alpha) d\alpha d\beta \\ V_6(t, r(t)) &= \int_{t-h}^t x^T(s) e^{k(s-t)} Q x(s) ds \end{aligned}$$

with

$$\begin{aligned} \eta &= [x^T(\beta - \tau(\beta)), y^T(\alpha)]^T \\ \xi(t) &= \begin{bmatrix} x^T(t), y^T(t), x^T(t - \tau(t)), \\ y^T(t - d), f^T(t), x^T(t - h) \end{bmatrix}^T \end{aligned}$$

Let  $L$  be the infinitesimal generator of random process. Then for each  $r(t) = i, i \in S$ , we have

$$LV(t, i) = \sum_{j=1}^6 LV_j(t, i) \quad (10)$$

where

$$\begin{aligned} LV_1(t, i) &\leq 2x^T(t)P_2A_ix(t) + 2x^T(t)P_2D_if(t) \\ &+ 2x^T(t)(F_i - P_2 + A^T_iP_3^T)y(t) - 2y^T(t)P_3y(t) \\ &+ 2x^T(t)P_2C_iy(t-d) + 2x^T(t)P_2B_ix(t-\tau(t)) \\ &+ 2y^T(t)P_3B_ix(t-\tau(t)) + 2y^T(t)P_3C_iy(t-d) \\ &+ 2y^T(t)P_3D_if(t) + x^T(t) \left( \sum_{j \neq i, j=1}^N \pi_{ij} F_j \right) x(t) \end{aligned}$$

$$\begin{aligned} LV_2(t, i) &\leq h y^T(t) e^{kh} S y(t) - \int_{t-\tau(t)}^t y^T(s) S y(s) ds \\ &- \int_{t-h}^{t-\tau(t)} y^T(s) S y(s) ds - k V_2(t, i) \end{aligned}$$

$$\begin{aligned} LV_3(t, i) &\leq h x^T(t - \tau(t)) R_{11} x(t - \tau(t)) \\ &+ 2x^T(t) R_{12}^T x(t - \tau(t)) + \int_{t-\tau(t)}^t y(s) R_{22} y(s) ds \\ &- 2x^T(t - \tau(t)) R_{12} x(t - \tau(t)) - k V_3(t, i) \end{aligned}$$

$$\begin{aligned} LV_4(t, i) &\leq y^T(t) M y(t) - y^T(t-d) e^{-kh} M y(t-d) \\ &- k V_4(t, i) \end{aligned}$$

$$\begin{aligned} LV_5(t, i) &\leq h \xi^T(t) e^{kh} W \xi(t) - \int_{t-\tau(t)}^t \xi^T(s) W \xi(s) ds \\ &- \int_{t-h}^{t-\tau(t)} \xi^T(s) W \xi(s) ds - k V_5(t, i) \end{aligned}$$

$$\begin{aligned} LV_6(t, i) &= x^T(t) Q x(t) - x^T(t-h) e^{-kh} Q x(t-h) \\ &- k V_6(t, i) \end{aligned}$$

Hence, we get

$$\begin{aligned} LV(t, i) &\leq \xi^T(t) \Xi_i \xi(t) - \int_{t-\tau(t)}^t q^T(s) \Phi_1 q(s) ds \\ &- \int_{t-h}^{t-\tau(t)} q^T(s) \Phi_2 q(s) ds - k V(t, i) \end{aligned} \quad (11)$$

where

$$\begin{aligned} \Xi_i &= T_i + \text{diag}\{\Psi_i, 0, 0, 0, 0, 0\} \\ \Psi_i &= \sum_{j \neq i, j \in S} \pi_{ij} F_j, q(t) = [\xi^T(s), y^T(s)]^T \\ \Phi_1 &= \begin{bmatrix} W & U \\ * & S - R_{22} \end{bmatrix}, \Phi_2 = \begin{bmatrix} W & V \\ * & S \end{bmatrix} \end{aligned}$$

From (7) and (8), we know that  $LV(t, i) \leq -kV(t, i)$  hold if  $\Xi_i < 0$ .

Consider  $\Psi_i \geq 0$  and  $1 \geq \sum_{i \neq j, j \in S^i} \pi_{ij} \geq 0$ , we can derive that  $LV(t, i) \leq -kV(t, i)$  holds if

$$\left( \sum_{i \neq j, j \in S^i} \pi_{ij} \right) T_i + \text{diag}\{\Psi_i, 0, 0, 0, 0, 0\} < 0 \quad (12)$$

By Schur complement, the above inequality can be rewritten as

$$\begin{aligned} &\begin{bmatrix} \left( \sum_{j \in S_{kn}^i} \pi_{ij} \right) T_i & T_{i1}^{2kn} \\ * & T_{i2}^{2kn} \end{bmatrix} \\ &+ \sum_{j \neq i, j \in S_{uk}^i} \pi_{ij} \begin{bmatrix} T_i & T_{i1j}^{2uk} \\ * & T_{i2j}^{2uk} \end{bmatrix} < 0 \end{aligned} \quad (13)$$

With ineq.(5) - ineq.(8), we can conclude that

$$LV(t, i) \leq -kV(t, i) \quad (14)$$

Hence

$$L(e^{kt} V(t, i)) = e^{kt} (LV(t, i) + kV(t, i)) \leq 0 \quad (15)$$

By Dynkins formula, one can obtain

$$\begin{aligned} EV(t, i) e^{kt} &= EV(t_0, i) e^{kt_0} + E \int_{t_0}^t L(e^{ks} V(s, i)) ds \\ &\leq EV(t_0, i) e^{kt_0} \end{aligned} \quad (16)$$

Hence

$$\lambda_{\min}(F_i) E \|x(t)\|^2 \leq EV(t, i) \leq EV(t_0, i) e^{-k(t-t_0)} \quad (17)$$

From the definition of  $V(t, i)$ , we know that

$$\begin{aligned} EV(t_0, i) &\leq [\lambda_{\max}(F_i) + h^2 \lambda_{\max}(S) + h^2 \lambda_{\max}(R) \\ &+ h \lambda_{\max}(M) + h^2 \lambda_{\max}(W) + h \lambda_{\max}(Q)] E \|\psi\|_h^2 \end{aligned} \quad (18)$$

where  $\|\psi\|_h = \sup_{-h \leq \theta \leq 0} \|\psi(\theta)\|$ .

Then, the following inequality is deduced

$$\mathbf{E} \|x(t)\| \leq ae^{-\frac{k}{2}(t-t_0)} \mathbf{E} \|\psi\|_h \quad (19)$$

where

$$a = \sqrt{\frac{\lambda_{\max}(F_i) + h^2 \lambda_{\max}(S) + h^2 \lambda_{\max}(R) + h \lambda_{\max}(M) + h^2 \lambda_{\max}(W) + h \lambda_{\max}(Q)}{\lambda_{\min}(F_i)}}$$

Finally, with Definition 1, we can conclude that MJNNS (1) is mean-square exponentially stable. The proof of Theorem 1 is thus completed.

Theorem 1 investigated MJNNS (1) in the situation that  $\pi_{ii}$  is unknown. Based on Theorem 1, the following theoretical result can be derived.

Corollary 1: When  $\pi_{ii}$  is known with given  $h$  and  $k$ , if there exist  $P_2, P_3, U, V, W$ , and symmetric positive-definite matrices  $F_i, R, M, S, Q$ , such that

$$\begin{bmatrix} \left(1 + \sum_{j \in S_{kn}^i} \pi_{ij}\right) \bar{T}_i & \bar{T}_{i1}^{kn} \\ * & \bar{T}_{i2}^{kn} \end{bmatrix} < 0 \quad (20)$$

$$\begin{bmatrix} \bar{T}_i & \bar{T}_{i1j}^{uk} \\ * & \bar{T}_{i2j}^{uk} \end{bmatrix} < 0, \forall j \in S_{uk}^i \quad (21)$$

$$\begin{bmatrix} W & U \\ * & S - R_{22} \end{bmatrix} > 0 \quad (22)$$

$$\begin{bmatrix} W & V \\ * & S \end{bmatrix} > 0 \quad (23)$$

where

$$\bar{\Sigma}_i = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \Sigma_{1,3} & \Sigma_{1,4} & \Sigma_{1,5} & \Sigma_{1,6} \\ * & \Sigma_{2,2} & \Sigma_{2,3} & \Sigma_{2,4} & \Sigma_{2,5} & \Sigma_{2,6} \\ * & * & \Sigma_{3,3} & 0 & 0 & 0 \\ * & * & * & \Sigma_{4,4} & 0 & 0 \\ * & * & * & * & \Sigma_{5,5} & 0 \\ * & * & * & * & * & \Sigma_{6,6} \end{bmatrix}$$

$$\Sigma_{1,1} = P_2 A_i + A_i^T P_2^T + \pi_{ii} F + Q_i + k F_i$$

$$\Sigma_{1,2} = F_i - P_2 + A_i^T P_3^T$$

$$\Sigma_{2,2} = h e^{kh} S - P_3 - P_3^T + M$$

$$\Sigma_{1,3} = P_2 B_i + R_{12}^T, \Sigma_{2,3} = P_3 B_i$$

$$\Sigma_{3,3} = h R_{11} - R_{12} - R_{12}^T, \Sigma_{1,4} = P_2 C_i$$

$$\Sigma_{2,4} = P_3 C_i, \Sigma_{4,4} = -M e^{-kh}$$

$$\Sigma_{1,5} = P_2 D_i + L^T K^T, \Sigma_{2,5} = P_3 D_i$$

$$\Sigma_{3,5} = N^T K^T, \Sigma_{5,5} = -2I, \Sigma_{6,6} = -Q e^{-kh}$$

$$\bar{T}_i = \bar{\Sigma}_i + \Omega + \Omega^T + h e^{kh} W$$

$$\bar{T}_{i1}^{kn} = \left[ \sum_{j \neq i, j \in S_{kn}^i} \pi_{ij} F_j, 0, 0, 0, 0, 0 \right]^T$$

$$\bar{T}_{i2}^{kn} = - \sum_{j \neq i, j \in S_{kn}^i} \pi_{ij} F_j, \bar{T}_{i1j}^{uk} = [F_j, 0, 0, 0, 0, 0]^T$$

$$\bar{T}_{i2j}^{uk} = -F_j, \Omega = [U, 0, -U + V, 0, 0, -V]$$

then, MJNNS (1) is mean-square exponentially stable.

Next, we consider the MJNNS with following nonlinear constraints

$$(f(t, z(t)) - K_1 z(t))^T (f(t, z(t)) - K_2 z(t)) \leq 0 \quad (24)$$

where  $K_2 > K_1$ .

Let  $F(t, z(t)) = f(t, z(t)) - K_1 z(t)$ , we have

$$F^T(t, z(t)) (F(t, z(t)) - K z(t)) \leq 0 \quad (25)$$

where  $K = K_2 - K_1$ .

The following inequality can be obtained

$$\begin{aligned} & -2F^T(t, z(t))F(t, z(t)) + 2F^T(t, z(t))K(Lx(t) \\ & + Nx(t - \tau(t))) \geq 0 \end{aligned} \quad (26)$$

Then MJNNS (1) can be represented as follows

$$\begin{cases} \dot{x}(t) = \bar{A}(r(t))x(t) + \bar{B}(r(t))x(t - \tau(t)) \\ \quad + C(r(t))\dot{x}(t - d) + D(r(t))F(t, z(t)) \\ z(t) = Lx(t) + Nx(t - \tau(t)) \\ x(t_0 + \theta) = \psi(\theta), \theta \in [-h, 0] \end{cases} \quad (27)$$

where  $\bar{A}(r(t)) = A(r(t)) + D(r(t))K_1 L$ ,  $\bar{B}(r(t)) = B(r(t)) + D(r(t))K_1 N$ .

It can be seen that system (1) and system (27) have the same structure with different system parameters. Hence based on Theorem 1, the following theoretical result can be derived.

**Theorem 2:** When  $\pi_{ii}$  is unknown with given  $h$  and  $k$ , if there exist  $P_2, P_3, U, V, W$ , and symmetric positive-definite matrices  $F_i, R, M, S, Q$ , such that

$$\begin{bmatrix} \left(\sum_{j \in S_{kn}^i} \pi_{ij}\right) T_i & T_{i1}^{kn} \\ * & T_{i2}^{kn} \end{bmatrix} < 0 \quad (28)$$

$$\begin{bmatrix} T_i & T_{i1j}^{uk} \\ * & T_{i2j}^{uk} \end{bmatrix} < 0, \forall j \neq i, j \in S_{uk}^i \quad (29)$$

$$\begin{bmatrix} W & U \\ * & S - R_{22} \end{bmatrix} > 0 \quad (30)$$

$$\begin{bmatrix} W & V \\ * & S \end{bmatrix} > 0 \quad (31)$$

where

$$\bar{\Sigma}_i = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \Sigma_{1,3} & \Sigma_{1,4} & \Sigma_{1,5} & \Sigma_{1,6} \\ * & \Sigma_{2,2} & \Sigma_{2,3} & \Sigma_{2,4} & \Sigma_{2,5} & \Sigma_{2,6} \\ * & * & \Sigma_{3,3} & 0 & 0 & 0 \\ * & * & * & \Sigma_{4,4} & 0 & 0 \\ * & * & * & * & \Sigma_{5,5} & 0 \\ * & * & * & * & * & \Sigma_{6,6} \end{bmatrix}$$

$$\begin{aligned}
 \Sigma_{1,1} &= P_2 (A_i + D_i K_1 L) + (A_i + D_i K_1 L)^T P_2^T \\
 &+ Q + k F_i, \Sigma_{1,2} = F_i - P_2 + (A_i + D_i K_1 L)^T P_3^T \\
 \Sigma_{2,2} &= h e^{kh} S - P_3 - P_3^T + M, \Sigma_{2,4} = P_3 C_i \\
 \Sigma_{1,3} &= P_2 (B_i + D_i K_1 N) + R_{12}^T \\
 \Sigma_{2,3} &= P_3 (B_i + D_i K_1 N), \Sigma_{1,4} = P_2 C_i \\
 \Sigma_{3,3} &= h R_{11} - R_{12} - R_{12}^T, \Sigma_{2,5} = P_3 D_i \\
 \Sigma_{1,5} &= P_2 D_i + L^T K^T, \Sigma_{3,5} = N^T K^T \\
 \Sigma_{4,4} &= -M e^{-kh}, \Sigma_{5,5} = -2I, \Sigma_{6,6} = -Q e^{-kh} \\
 T_i &= \Sigma_i + \Omega + \Omega^T + h e^{kh} W \\
 T_{i1}^{kn} &= \left[ \sum_{j \in S_{kn}^i} \pi_{ij} F_j, 0, 0, 0, 0, 0 \right]^T \\
 T_{i2}^{kn} &= - \sum_{j \in S_{kn}^i} \pi_{ij} F_j, T_{i1j}^{uk} = [F_j, 0, 0, 0, 0, 0]^T \\
 T_{i2j}^{uk} &= -F_j, \Omega = [U, 0, -U + V, 0, 0, -V]
 \end{aligned}$$

then, MJNNS (27) is mean-square exponentially stable.

Proof: The proof is similar to that of Theorem 1, and thus is omitted.

Theorem 2 studied MJNNS (27) in the situation that  $\pi_{ii}$  is unknown. Based on Theorem 2, the following theoretical result can be derived.

Corollary 2: When  $\pi_{ii}$  is known with given  $h$  and  $k$ , if there exist  $P_2, P_3, U, V, W$ , and symmetric positive-definite matrices  $F_i, R, M, S, Q$ , such that

$$\begin{bmatrix} \left(1 + \sum_{j \in S_{kn}^i} \pi_{ij}\right) \bar{T}_i & \bar{T}_{i1}^{kn} \\ * & \bar{T}_{i2}^{kn} \end{bmatrix} < 0 \quad (32)$$

$$\begin{bmatrix} \bar{T}_i & \bar{T}_{i1j}^{uk} \\ * & \bar{T}_{i2j}^{uk} \end{bmatrix} < 0, \forall j \in S_{uk}^i \quad (33)$$

$$\begin{bmatrix} W & U \\ * & S - R_{22} \end{bmatrix} > 0 \quad (34)$$

$$\begin{bmatrix} W & V \\ * & S \end{bmatrix} > 0 \quad (35)$$

where

$$\bar{\Sigma}_i = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \Sigma_{1,3} & \Sigma_{1,4} & \Sigma_{1,5} & \Sigma_{1,6} \\ * & \Sigma_{2,2} & \Sigma_{2,3} & \Sigma_{2,4} & \Sigma_{2,5} & \Sigma_{2,6} \\ * & * & \Sigma_{3,3} & 0 & 0 & 0 \\ * & * & * & \Sigma_{4,4} & 0 & 0 \\ * & * & * & * & \Sigma_{5,5} & 0 \\ * & * & * & * & * & \Sigma_{6,6} \end{bmatrix}$$

$$\begin{aligned}
 \Sigma_{1,1} &= P_2 (A_i + D_i K_1 L) + (A_i + D_i K_1 L)^T P_2^T \\
 &+ \pi_{ii} F_i + Q + k F_i, \Sigma_{2,3} = P_3 (B_i + D_i K_1 N) \\
 \Sigma_{1,2} &= F_i - P_2 + (A_i + D_i K_1 L)^T P_3^T \\
 \Sigma_{2,2} &= h e^{kh} S - P_3 - P_3^T + M, \Sigma_{1,4} = P_2 C_i \\
 \Sigma_{1,3} &= P_2 (B_i + D_i K_1 N) + R_{12}^T, \Sigma_{2,4} = P_3 C_i \\
 \Sigma_{3,3} &= h R_{11} - R_{12} - R_{12}^T, \Sigma_{4,4} = -M e^{-kh} \\
 \Sigma_{1,5} &= P_2 D_i + L^T K^T, \Sigma_{2,5} = P_3 D_i \\
 \Sigma_{3,5} &= N^T K^T, \Sigma_{5,5} = -2I, \Sigma_{6,6} = -Q e^{-kh} \\
 \bar{T}_i &= \bar{\Sigma}_i + \Omega + \Omega^T + h e^{kh} W \\
 \bar{T}_{i1}^{kn} &= \left[ \sum_{j \neq i, j \in S_{kn}^i} \pi_{ij} F_j, 0, 0, 0, 0, 0 \right]^T \\
 \bar{T}_{i2}^{kn} &= - \sum_{j \neq i, j \in S_{kn}^i} \pi_{ij} F_j, \bar{T}_{i1j}^{uk} = [F_j, 0, 0, 0, 0, 0]^T \\
 \bar{T}_{i2j}^{uk} &= -F_j, \Omega = [U, 0, -U + V, 0, 0, -V]
 \end{aligned}$$

then, MJNNS (27) is mean-square exponentially stable.

## 4 Simulation Results

In this section, we will verify the proposed methodology by giving an illustrative example. The MJNNS with following system parameters is included

$$\begin{aligned}
 A_1 &= \begin{bmatrix} -1 & 0.4 \\ 0.4 & -1.2 \end{bmatrix}, B_1 = \begin{bmatrix} -0.4 & 0 \\ 0.3 & -0.4 \end{bmatrix} \\
 C_1 &= \begin{bmatrix} 0.2 & 0 \\ -0.1 & 0.2 \end{bmatrix}, D_1 = \begin{bmatrix} -0.4 & 0.3 \\ 0 & -0.3 \end{bmatrix} \\
 A_2 &= \begin{bmatrix} -1 & 0.5 \\ 0.4 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} -0.2 & 0 \\ 0.3 & -0.3 \end{bmatrix} \\
 C_2 &= \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0 & -0.4 \end{bmatrix} \\
 A_3 &= \begin{bmatrix} -1 & 0.4 \\ 0.3 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} -0.5 & 0.1 \\ 0.4 & -0.3 \end{bmatrix} \\
 C_3 &= \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.1 \end{bmatrix}, D_3 = \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & -0.2 \end{bmatrix} \\
 L &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, N = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix} \\
 \tau(t) &= 0.6 + 0.1 \sin^2(15t), k = 0.1, d = 0.4 \\
 f(t, z(t)) &= (|z(t) + 1| - |z(t) - 1|)/2, K = I
 \end{aligned}$$

First we consider the given MJNNS with transition rates as follow

$$\begin{aligned}
 \Pi_1 &= \begin{bmatrix} -0.5 & 0.3 & 0.2 \\ 0.3 & -0.9 & 0.6 \\ 0.2 & 0.2 & -0.4 \end{bmatrix} \\
 \Pi_2 &= \begin{bmatrix} -0.7 & - & - \\ 0.4 & -0.5 & 0.1 \\ 0.1 & 0.3 & -0.4 \end{bmatrix} \\
 \Pi_3 &= \begin{bmatrix} -0.5 & - & - \\ - & -0.3 & - \\ 0.4 & 0.2 & -0.6 \end{bmatrix}
 \end{aligned}$$

According to the derived stability criteria, Table 1 can be obtained.

Table 1: The maximum stable delay of the MJNNS with different transition rates

transition rates	case 1	case 2	case 3
$h$	1.1434	1.0093	0.9225

Next, we consider the given MJNNS with transition rates that are unknown, partially known and completely known as follow

$$\begin{aligned} \Pi_4 &= \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \\ \Pi_5 &= \begin{bmatrix} - & - & - \\ 0.2 & -0.4 & 0.2 \\ 0.2 & 0.3 & -0.5 \end{bmatrix} \\ \Pi_6 &= \begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.2 & -0.4 & 0.2 \\ 0.2 & 0.3 & -0.5 \end{bmatrix} \end{aligned}$$

According to the presented theoretical results, Table 2 can be obtained.

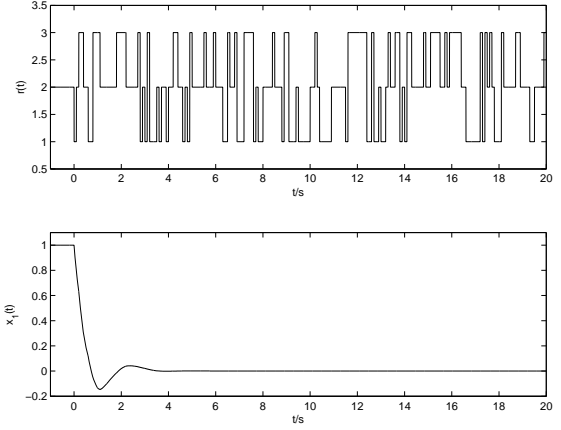
Table 2: The maximum stable delay of the MJNNS with transition rates unknown, partly known and fully known

trans. rates	unknown	par. known	ful. known
$h$	0.7507	0.8149	1.1524

**Remark 2:** Table 1 shows the maximum stable delay of the MJNNS changes with different transition rates. This means that transition rates have an effect on the stochastic stability of the MJNNS. Table 2 displays the relationship between the maximum stable delay of the MJNNS and transition rates that are unknown, partially known and fully known, respectively. It can be seen that the biggest maximum stable delay occurs in the situation that transition rates are fully known for the best utilization of all commutation information. In addition, it is illustrated that no matter how many transition rates are accessible, the presented stability criteria are still available.

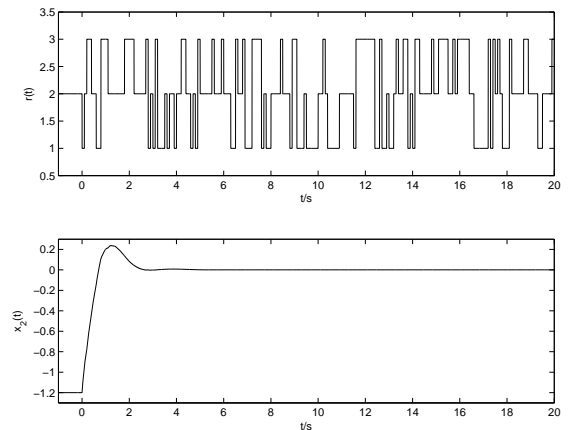
Next, the numerical simulation is carried out to analyze the dynamics of the MJNNS. The numerical simulation is with the initial state value  $\psi(\theta) = [1, -1.2]^T$ ,  $\theta \in (-1, 0)$ ,  $r(t_0) = 2$ , the simulation step 0.001 second, the transition rates  $\Pi_6$  and the switching time 0.1 second. Corresponding simulation results are shown in Figs. 1 and 2.

**Remark 3:** Fig. 1 depicts the time response of the state variable of the MJNNS. Fig. 2 depicts the time


 Figure 1: Time response of the state variable  $x_1(t)$  of the MJNNS.

response of the state variable of the MJNNS. It can be seen that the MJNNS is mean-square exponential stable, and spends about 6 seconds converging to the zero state.

**Remark 4:** For the given example,  $\dot{\tau}(t) = 1.5 \sin(30t)$ , and the maximum upper bound of  $\dot{\tau}(t)$  is 1.5, greater than 1. Based on the constructed Lyapunov function, the derived stochastic stability criteria have no constraints on the upper bound of  $\dot{\tau}(t)$ . In addition, the calculated maximum stable delay of the MJNNS is 1.1524, bigger than the real maximum delays upper bound 0.7, which means that this system should be mean-square exponentially stable, and the simulation results in Figs. 1 and 2 accord with the proposed stability criteria.


 Figure 2: Time response of the state variable  $x_2(t)$  of the MJNNS.

## 5 Conclusions

In this paper, the stochastic stability analysis on Markovian jump neutral nonlinear systems with time-varying delay is concerned. By constructing a novel Lyapunov function, the mean-square exponential stability criterion, independent of time-varying delays derivative, has been concluded for the circumstance that transition rates are completely accessible, partially accessible and non-accessible. Subsequently, the developed stability criterion has been extended to the systems with a different bounded sector nonlinear constrain. At last, some numerical examples have been included to illustrate the effectiveness of the obtained theoretical results.

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